ON THE CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS

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1. **Introduction.** Let S be a sequence $\{U_n(x)\}(n=1, 2, \cdots)$ of linear operations defined over the elements $\{x\}$ of a Banach space[1] E with values lying in a second B space E_1. The fundamental convergence theorem associated with such sequences may be stated as follows:

BANACH-STEINHAUS THEOREM. *For the convergence of S over E it is necessary and sufficient that S converge over a set H dense in a sphere K of E and that the set of norms $\{\|U_n\|\}$ be bounded.*

The solution of convergence problems involving sequences of positive functionals defined on certain function-spaces[2] led the author to investigate a different form of this convergence criterion which seems to have a more direct practical application in some special cases of importance. The development hinges on the fact that from a certain point of view every linear operation has domains over which it is "positive."

2. **A new convergence criterion.** Let us make the following definition:

DEFINITION. *A set of points P in a normed vector space will be called a p-set provided that, for every pair of points z and z' of P, $\|z-z'\| \leqq \|z+z'\|$.*

Such a set has many of the characteristic properties of a set of real numbers all of the same sign. In effect, we have extended the notion of "positiveness" to the elements of an abstract space.

In terms of this definition we may now state the following theorem:

THEOREM 1. *For the convergence of S over E, it is sufficient that there exist a sphere K of E such that* (a) *S converges over a set H dense in K and* (b) *for all fixed n sufficiently large, the set of transforms $\{U_n(x)\}$ $(x \subset K)$, is a p-set, P_n.*

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[1] A space of type B. Cf. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 53. In general we follow the terminology of this treatise.

[2] R. P. Bailey, *Convergence of sequences of positive linear functional operations*, Duke Math. J. vol. 2 (1936) pp. 287–303.

Assuming the conditions of Theorem 1 are satisfied, it is possible to determine a sphere $K' \subset K$ of center $\xi \subset H$ and radius $r > 0$. Let $\xi' = \xi/r$. Then, if $\|x\| \leqq 1$,

$$\|\xi - r(\xi' \pm x)\| = \|rx\| \leqq r,$$

so that $r(\xi' \pm x) \subset K' \subset K$. Hence, by our hypothesis, there exists a positive integer N, such that, for $n \geqq N$ and $\|x\| \leqq 1$,

$$P_n \supset U_n[r(\xi' \pm x)] = [U_n(r\xi') \pm U_n(rx)].$$

Setting $z = U_n(r\xi') + U_n(rx)$ and $z' = U_n(r\xi') - U_n(rx)$, since $\|z - z'\| \leqq \|z + z'\|$, we may conclude

$$\|U_n(x)\| \leqq \|U_n(\xi')\|, \qquad \|x\| \leqq 1, n \geqq N.$$

Thus $|U_n| \leqq \|U_n(\xi')\|$ $(n \geqq N)$.

But the sequence S converges at ξ and hence at ξ'. It follows that the set $\{\|U_n(\xi')\|\}$ $(n = 1, 2, \cdots)$ is bounded, and therefore the set of norms $\{|U_n|\}$ is bounded. Thus the Banach-Steinhaus conditions are satisfied, and S converges at all points of E. This completes the proof of Theorem 1.

As an illustration consider the sequences of so-called "positive" linear functional operations, that is, linear operations which associate real numbers $U(x)$ with the elements $x = x(t)$ of a function-space in such a way that $U(x) \geqq 0$ whenever $x(t) \geqq 0$. For instance let $\{\alpha_n(t)\}$ $(n = 1, 2, \cdots)$ be a set of monotone non-decreasing functions defined on $(0, 1)$. The functionals $U_n(x) = \int_0^1 x(t)d\alpha_n(t)$ $(n = 1, 2, \cdots)$ are positive linear functionals over the B space C consisting of the continuous functions $\{x(t)\}$ defined on $(0, 1)$ with norm $\|x\| = \max_{0 \leqq t \leqq 1} |x(t)|$, and therefore obviously transform spheres of non-negative functions into the p-set consisting of the positive real numbers. It follows that the sequence $\{U_n(x)\}$ cannot converge over a set dense in a sphere of C without converging over the whole of C.

It should not be assumed, however, that the only practical value of the criterion lies in its application to sequences of positive functionals. All of the better known Banach spaces contain easily recognized p-sets of a very extensive character. We may point, for instance, to the set of non-negative functions of C space, the set of points $x = (x_1, x_2, \cdots, x_n)$ of Euclidean n-space such that $x_i \geqq 0$ $(i = 1, 2, \cdots, n)$, and the set of functions $\{x(t)\}$ Lebesgue square integrable on $(0, 1)$ with the norm $\|x\| = [\int_0^1 x^2 dt]^{1/2}$ which consists of those functions having non-negative Fourier coefficients. In fact, we shall prove in what follows that the sphere with arbitrary center z_0 and radius $\|z_0\|/2$ in any Banach space is a p-set.

In other words, the real value of the criterion lies in the fact that it is frequently obvious at a glance that certain spheres in E transform into p-sets in E_1, while the boundedness of the norms may be difficult to establish.

3. **Necessity of the criterion.** In an abstract of this paper[3] the author made an error in stating that the conditions of Theorem 1 were necessary for convergence, as well as sufficient. The question of necessity is clarified in this section. We shall assume the Banach-Steinhaus conditions are satisfied, and ask under what circumstances it is necessary that some sphere of E transform into p-sets in E_1.

By hypothesis there exists a number $M>0$ such that $|U_n| \leqq M$ for all n, and S converges at all points of E to a linear operation $U(x)$. It is convenient to divide the discussion into two parts, depending upon whether or not $U(x)$ vanishes identically over E. Denote by θ_1 the zero element of E_1.

Case 1. ($U(x) \not\equiv \theta_1$.) Choose $\xi \subset E$ such that $U(\xi) \neq \theta_1$. Since $\|U(\xi)\| > 0$, positive numbers $\epsilon > 0$, $r > 0$ may be chosen such that $\epsilon + rM \leqq \|U(\xi)\|/2$. If x is any point of the sphere K defined by the condition $\|\xi - x\| \leqq r$, and N be so chosen that $\|U(\xi) - U_n(\xi)\| \leqq \epsilon$ ($n \geqq N$), it follows that for $n \geqq N$ and $\|\xi - x\| \leqq r$,

$$\begin{aligned} \|U(\xi) - U_n(x)\| &\leqq \|U(\xi) - U_n(\xi)\| + \|U_n(\xi - x)\| \\ &\leqq \epsilon + rM \\ &\leqq \|U(\xi)\|/2. \end{aligned}$$

That is, for $n \geqq N$, and $\|\xi - x\| \leqq r$, all transforms $\{z = U_n(x)\}$ lie in a sphere of E_1 defined by the condition

$$\|U(\xi) - z\| \leqq \|U(\xi)\|/2.$$

Moreover this spherical set is a p-set, for if z and z' are any two of its points,

$$\begin{aligned} \|z - z'\| &\leqq \|z - U(\xi)\| + \|U(\xi) - z'\| \\ &\leqq \|U(\xi)\|, \end{aligned}$$

while

$$\begin{aligned} \|z + z'\| &= \|z - U(\xi) + z' - U(\xi) + 2U(\xi)\| \\ &\geqq 2\|U(\xi)\| - \|z - U(\xi)\| - \|z' - U(\xi)\| \\ &\geqq 2\|U(\xi)\| - \|U(\xi)\| \\ &\geqq \|U(\xi)\|. \end{aligned}$$

[3] Bull. Amer. Math. Soc. abstract 43-3-97.

That is, the criterion of Theorem 1 is satisfied. We have proved this theorem:

THEOREM 2. *The conditions of Theorem* 1 *are necessary for the convergence of S over E to a linear operation $U(x)$ which does not vanish identically.*

Case 2. ($U(x)\equiv\theta_1$.) An example will be furnished to show that no necessity exists in this case. Let E consist of the set of continuous functions $\{x(t)\}$ defined on (0, 1) with $x(0)=0$. We use the customary norm $\|x\|=\max_{0\leq t\leq 1}|x(t)|$. If $U_n(x)=n\int_0^{1/n}x(t)dt$ $(n=1, 2, \cdots)$, clearly $\lim_{n\to\infty}U_n(x)=0$ at all points of E. Nevertheless, it is impossible to choose $N>0$, and a sphere K of E, such that for fixed $n\geqq N$ and $x\subset K$, the transforms $\{U_n(x)\}$ constitute p-sets P_n. For, suppose K be any arbitrary sphere $\|x_0-x\|\leqq r$ with center x_0. Since $x_0(0)=0$, it is possible to determine a point t_0 $(0<t_0<1)$ such that $\max|x_0(t)|$ $(0\leqq t\leqq t_0)$ is arbitrarily small. Clearly, if t_0 is sufficiently small, there exist functions $x_1(t)$ and $x_2(t)$ having the following properties:

(i) $$x_1(t)>0, \qquad x_2(t)<0, \qquad 0<t\leqq t_0,$$

(ii) $$|x_0(t)-x_1(t)|\leqq r, \qquad |x_0(t)-x_2(t)|\leqq r, \qquad 0\leqq t\leqq 1.$$

But now, if $n\geqq 1/t_0$, $U_n(x_1)>0$, while $U_n(x_2)<0$. The set of transforms $\{U_n(x)\}$ $(x\subset K)$ therefore cannot form a p-set, no matter what sphere K is selected, if n is sufficiently large.

4. **Convergence on a locally compact B space.** We have shown in the preceding section that the conditions of Theorem 1 need not be satisfied in all cases to obtain convergence. However, it is possible to establish the following criterion:

THEOREM 3. *For the convergence of S over a B space E which is locally compact, the conditions of Theorem* 1 *are necessary and sufficient.*[4]

We have only to demonstrate necessity in the case $U(x)\equiv\theta_1$. The remainder follows at once from Theorems 1 and 2. We shall need the following lemma.

LEMMA. *A sequence $\{F_n(x)\}$ $(n=1, 2, \cdots)$ of linear operations defined over a locally compact B space E may not possess both of the following characteristics*:

(i) $$|F_n|=1, \qquad n=1, 2, \cdots,$$

(ii) $$\liminf_{n\to\infty}\|F_n(x)\|=0 \qquad \textit{for all } x \textit{ of } E.^5$$

[4] We may mention in passing that the property of local compactness holds if, and only if, the space has a finite number of dimensions.

[5] Note that the operations $U_n(x)\equiv n\int_0^{1/n}x(t)dt$ defined in §3 do possess both properties.

Suppose $\{G_n(x)\}$ $(n=1, 2, \cdots)$ is any infinite selection of the operations $\{F_n(x)\}$. Let $\epsilon>0$ be any arbitrarily specified real positive number. For each $n=1, 2, \cdots$ there exists a point $\xi_n \subset E$ such that $\|\xi_n\| \leqq 1$ and $\|G_n(\xi_n)\| \geqq 1-\epsilon$. Since the unit sphere is compact, it is possible to extract from the sequence $\{\xi_n\}$ a subsequence $\{\xi_{n_k}\}$ which converges to a point ξ of E. For n_k sufficiently large, $\|\xi-\xi_{n_k}\| \leqq \epsilon$. Hence, for all n_k sufficiently large,

$$\begin{aligned}\|G_{n_k}(\xi)\| &= \|G_{n_k}(\xi_{n_k}) + G_{n_k}(\xi - \xi_{n_k})\| \\ &\geqq \|G_{n_k}(\xi_{n_k})\| - \|G_{n_k}(\xi - \xi_{n_k})\| \\ &\geqq 1 - 2\epsilon,\end{aligned}$$

and therefore $\limsup_{n\to\infty} \|G_n(\xi)\| \geqq 1$. Since $\{G_n(x)\}$ is any arbitrary infinite selection of the operations $\{F_n(x)\}$, this contradicts (ii).

Now consider any arbitrary sequence $\{U_n(x)\}$ defined over a locally compact space E, such that $\lim_{n\to\infty} U_n(x) = U(x)$ vanishes at all points of E. We wish to show that there exists a sphere K of E such that $\{U_n(x)\}$ $(x \subset K)$ constitutes a p-set P_n, for all n sufficiently large.

The contradomain of an operation $U_n(x) \equiv \theta_1$ $(x \subset E)$ is obviously a p-set. Hence consider only the subsequence $\{U_{n_k}(x)\}$ of $\{U_n(x)\}$ consisting of those operations which do not vanish identically over[6] E. Since $|U_{n_k}| > 0$, we may set

$$F_{n_k}(x) = |U_{n_k}|^{-1} \cdot U_{n_k}(x), \qquad x \subset E.$$

Since $|F_{n_k}| = 1$ $(k=1, 2, \cdots)$, it is impossible that $\liminf_{k\to\infty} \|F_{n_k}(x)\| = 0$ at all points of E, by the previously proved lemma. Hence there exists a point $\xi \neq \theta$, and a number $r>0$, such that

$$\liminf_{k\to\infty} |U_{n_k}|^{-1} \cdot \|U_{n_k}(\xi)\| > 2r > 0.$$

Therefore it is possible to fix N so large, that for $n_k > N$,

$$|U_{n_k}|^{-1} \cdot \|U_{n_k}(\xi)\| \geqq 2r,$$

or

$$r|U_{n_k}| \leqq \|U_{n_k}(\xi)\|/2.$$

Now, if $\|x-\xi\| \leqq r$, and $n_k > N$,

$$\|U_{n_k}(x) - U_{n_k}(\xi)\| \leqq r|U_{n_k}| \leqq \|U_{n_k}(\xi)\|/2.$$

[6] We assume the subsequence is infinite; in the contrary case nothing remains to be proved.

Since $\|U_{n_k}(\xi)\| > 0$, the sphere $\|z - U_{n_k}(\xi)\| \leqq \|U_{n_k}(\xi)\|/2$, is non-vacuous. That such a sphere is a p-set was demonstrated in §3. The sphere K we were required to construct has therefore been shown to exist, and Theorem 3 is proved.

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ON THE APPROXIMATION OF FUNCTIONS BY SUMS OF ORTHONORMAL FUNCTIONS

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1. **Introduction.** The main object of this paper is to derive, in a simple manner, upper bounds for the norms of the derivative of

$$\sum_{i=0}^{n} a_i \phi_i(x) \tag{1}$$

in C and L^2 spaces, where the a_i are arbitrary constants, and $\{\phi_i(x)\}$ is any set of functions on a given finite or infinite interval (a, b). We apply our method, properly modified, first to the case where the $\phi_i(x)$ are characteristic solutions of conjugate sets of integral equations, then to other classes of functions whose first derivatives $\{\phi_i'(x)\}$ are orthogonal with respect to a weight function $\sigma(x)$. Finally, we apply our results to the question of convergence of sums[1] of type (1) that minimize

$$\int_a^b \rho(x) \left| f(x) - \sum_{i=0}^{n} a_i \phi_i(x) \right|^m dx, \qquad m > 0.$$

The leading results of our investigation may be summarized briefly as follows:

$$\text{(A)} \qquad \left| \frac{d}{dx} \sum_{i=0}^{n} a_i \phi_i(x) \right| \leqq \lambda_n k(x) \left(\int_a^b [\sum_{i=0}^{n} a_i \phi_i(x)]^2 dx \right)^{1/2}$$

where λ_n is a positive number that increases with n and $k(x)$ is a func-

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[1] For the specialized cases when the approximating functions are trigonometric sums or polynomials, see D. Jackson, *The theory of approximation*, Amer. Math. Soc. Colloquium Publications vol. 11, 1930, pp. 86–89, 96–101.

tion, independent of n, that is less than or equal to a bound M almost everywhere.

$$\text{(B)} \qquad \left(\int_a^b \sigma(x)\left[\frac{d}{dx}\sum_{i=0}^{n} a_i\phi_i(x)\right]^2 dx\right)^{1/2} \leqq N_n\left(\int_a^b [\sum_{i=0}^{n} a_i\psi_i(x)]^2 dx\right)^{1/2}$$

where $\{\psi_i(x)\}$ may be *any* set of orthonormal functions on (a, b), and N_n, for a given n and arbitrary coefficients a_i, is the least number that relates the left- and right-hand sides of (B).

Somewhat similar investigations have been made by Hille, Szegö, and Tamarkin,[2] McEwen,[3] and Shohat[4] but in each case with more restricted functions and by methods that are different from ours.

2. **Rough bounds for the derivative of sums of characteristic solutions.** Let $K(x, t)$ be a given continuous function on an interval $a \leqq x \leqq b$, $a \leqq \tau \leqq b$, and let

$$(2) \qquad \phi(x) = \lambda\int_a^b K(x, t)\psi(t)dt, \qquad \psi(x) = \lambda\int_a^b K(t, x)\phi(t)dt$$

be conjugate integral equations which have $K(x, t)$ as kernel. It is well known[5] that, except for the special case when $K(x, t) = \sum_{i=0}^{n}\phi_i(x)\psi_i(t)/\lambda_i$, there exists an infinite set of characteristic values for λ, and corresponding sets of characteristic functions,

$$\phi_0(x), \phi_1(x), \cdots, \phi_n(x), \cdots, \psi_0(x), \psi_1(x), \cdots, \psi_n(x), \cdots$$

that satisfy these equations. The set of functions $\{\psi_i(x)\}$ can be adjusted so that all of the characteristic numbers are positive and it can be assumed without loss of generality that $\lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n \leqq \cdots$. Moreover, the sets $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ can be regarded as separately orthonormal on (a, b). In addition to the hypotheses of continuity on $K(x, t)$ we assume differentiability with respect to x under the integral sign of $\int_a^b K(x, t)dt$ and that

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[2] Hille, Szegö, and Tamarkin, *On some generalizations of a theorem of A. Markoff*, Duke Math. J. vol. 3 (1937) pp. 729–739.

[3] W. H. McEwen, *A note on an extension of Bernstein's theorem*. Amer. J. Math. vol. 60 (1938) pp. 309–319.

[4] J. Shohat. *On a general formula in the theory of Tschebycheff polynomials and its application*, Trans. Amer. Math. Soc. vol. 29 (1927), p. 569.

[5] See E. Schmidt. *Zur theorie der linearen und nichtlinearen Integralgleichungen*, Math. Ann. vol. 63 (1907) pp. 459–463.

$$\int_a^b \left[\frac{\partial}{\partial x} K(x, t)\right]^2 dt$$

as a function of x exists almost everywhere on (a, b).

Let $a_0, a_1, \cdots, a_n$ be any set of coefficients and let

$$s_n(x) = \sum_{i=0}^{n} a_i\phi_i(x), \qquad \bar{s}_n(x) = \sum_{i=0}^{n} a_i\psi_i(x), \qquad \bar{s}_{n\lambda}(x) = \sum_{i=0}^{n} a_i\lambda_i\psi_i(x).$$

From (2) it follows that

$$\frac{d}{dx} s_n(x) = \int_a^b \frac{\partial}{\partial x} K(x, t)\bar{s}_{n\lambda}(t)dt$$

whence, as a consequence of Schwarz's inequality, we have

$$\left|\frac{d}{dx} s_n(x)\right| \leqq \left(\int_a^b \left[\frac{\partial}{\partial x} K(x, t)\right]^2 dt\right)^{1/2} \left(\int_a^b \bar{s}_{n\lambda}^2(t)dt\right)^{1/2}.$$

By the property of orthonormality of the set $\{\psi_i(x)\}$, and the fact that the λ's increase with n,

$$\int_a^b \bar{s}_{n\lambda}^2(t)dt = \sum_{i=0}^{n} a_i^2\lambda_i^2 \leqq \lambda_n^2 \sum_{i=0}^{n} a_i^2 = \lambda_n^2 \int_a^b \bar{s}_n^2(t)dt.$$

The above relation, in view of the fact that the a's are arbitrary, is the best possible bound for $\int_a^b \bar{s}_{n\lambda}^2(t)dt$ in terms of $\int_a^b \bar{s}_n^2(t)dt$ since the equality holds when a_n is the only nonzero coefficient. However, the last integral may be replaced by the integral of the square of a sum of any $n+1$ orthonormal functions on the interval (a, b) if the coefficients of this sum are $a_0, a_1, \cdots, a_n$. In particular, $\int_a^b \bar{s}_n^2(t)dt$ may be replaced by $\int_a^b s_n^2(t)dt$.

The previous discussion may be summarized as follows.

THEOREM A. *If the kernel $K(x, t)$ satisfies the hypotheses of the preceding discussion, and if $s_n(x)$ is any arbitrary sum of $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$, then*

$$\left|\frac{d}{dx} s_n(x)\right| \leqq M\lambda_n \left(\int_a^b s_n^2(t)dt\right)^{1/2}$$

where M is a constant, over the parts of the interval (a, b) for which $\int_a^b [\partial K(x, t)/\partial x]^2 dt$ is bounded. Moreover, if (a, b) is finite and u_n is the maximum of $|s_n(x)|$ on (a, b), then

$$\left|\frac{d}{dx} s_n(x)\right| \leqq M'\lambda_n u_n, \qquad M' = M(b - a)^{1/2}. \tag{3}$$

For special sets of characteristic functions and kernels, closer bounds then (3) are obtainable by other means[6] than ours. Thus, for example, if we consider the particular kernel[7]

$$K(x, t) = x(1 - t), \qquad 0 \leqq x \leqq t,$$
$$= t(1 - x), \qquad t \leqq x \leqq 1,$$

on the interval $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, the integral equation $u(x) = \lambda \int_0^1 K(x, t)u(t)dt$ has as characteristic solutions and corresponding characteristic numbers the sets $\{\sin n\pi x\}$ and $\{n^2\pi^2\}$, respectively. The application of (3) gives a bound of order n^2u_n for the derivative of a sum of sine functions, which is rough when compared with the well known Bernstein's theorem for trigonometric sums.[8] However, the bound given by (3) is of interest since it applies to a very wide class of functions for which it seems impossible to obtain better results unless more restrictive hypotheses are assumed on the kernel or on the characteristic functions themselves.

3. **Bernstein's theorem in L^2 space for a sum of functions whose derivatives are orthogonal with respect to a weight function.** Hille, Szegö, and Tamarkin[9] have established the corresponding Markoff's theorem for L^m spaces when $m \geqq 1$, but only when the sums involved are polynomials or trigonometric sums. Our method of proof is entirely different from theirs and we are not able to obtain the results of this section from their paper.

Let $\{\phi_i(x)\}$ be any set of functions, not necessarily orthonormal, but let the set of first derivatives of these functions be orthogonal with respect to a positive weight function $\sigma(x)$ on the interval (a, b). Assume that the normalization factor $N_i = (\int_a^b \sigma(x)[\phi_i'(x)]^2 dx)^{1/2}$ increases with the subscript i. Let $K_n(x, t) = \sum_{i=0}^{n} \phi_i(x)\psi_i(t)$ where $\{\psi_i(x)\}$ is any set of orthonormal functions on (a, b). Let $s_n(x)$ denote any arbitrary sum of the functions $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$, and let $s_n'(x)$ be its first derivative. Let $\bar{s}_n(x) = \sum_{i=0}^{n} a_i\psi_i(x)$ where $a_0, a_1, \cdots, a_n$ are the same coefficients as in $s_n(x)$.

Since

$$s_n'(x) = \int_a^b \frac{\partial}{\partial x} K_n(x, t)\bar{s}_n(t)dt$$

[6] See McEwen, op. cit. pp. 295–296.

[7] The kernel $K(x, t)$ in this instance is the Green's function of the system $d^2u/dx^2 + \lambda u = 0$, $u(0) = u(1) = 0$, and thus highly restricted when compared with any kernel of a system of form (2).

[8] See Jackson, op. cit. p. 80.

[9] Hille, Szegö, and Tamarkin, op. cit. Footnote 2.

it follows from the hypotheses placed on the sets $\{\phi_i'(x)\}$, $\{N_i\}$, and $\{\psi_i(x)\}$ that if this equation is multiplied by $\sigma(x)s_n'(x)$ and integrated over (a, b), then

$$\int_a^b \sigma(x)[s_n'(x)]^2dx = \int_a^b \left[\int_a^b \sigma(x)s_n'(x)K_n'(x,t)dx\right]\bar{s}_n(t)dt$$

$$= \int_a^b \left[\sum_{i=0}^n a_i N_i^2\psi_i(t)\right]\left[\sum_{i=0}^n a_i\psi_i(t)\right]dt$$

$$= \sum_{i=0}^n a_i^2N_i^2 \leqq N_n^2\sum_{i=0}^n a_i^2 = N_n^2\int_a^b \bar{s}_n^2(t)dt.$$

The factor N_n^2 is obviously the least number that relates the right- and left-hand members of the above inequality since the equality holds when all but a_n of the coefficients $a_1, a_2, \cdots, a_n$ are equal to zero. As a result we have this theorem.

THEOREM B. *If $\{\phi_i(x)\}$ is a given set of functions, and if there exists a positive weight function $\sigma(x)$ such that the set of functions*

$$\sigma^{1/2}(x)\phi_0'(x), \sigma^{1/2}(x)\phi_1'(x), \cdots, \sigma^{1/2}(x)\phi_n'(x)$$

are orthogonal on (a, b) and if $s_n(x) = \sum_{i=0}^n a_i\phi_i(x)$ is any arbitrary sum of the set $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$ and $\bar{s}_n(x) = \sum_{i=0}^n a_i\psi_i(x)$ is the sum of any arbitrary set of orthonormal functions on the interval (a, b), the coefficients $a_0, a_1, \cdots, a_n$ being the same in both $s_n(x)$ and $\bar{s}_n(x)$, then

$$\left(\int_a^b \sigma(x)[s_n'(x)]^2dx\right)^{1/2} \leqq N_n\left(\int_a^b [\bar{s}_n(x)]^2dx\right)^{1/2} \tag{4}$$

gives the closest relation between the integrals

$$\left(\int_a^b \sigma(x)[s_n'(x)]^2dx\right)^{1/2} \text{ and } \left(\int_a^b [\bar{s}_n(x)]^2dx\right)^{1/2} \text{ when}$$

$$N_n = \left(\int_a^b \sigma(x)[\phi_n'(x)]^2dx\right)^{1/2}.$$

As a first application of the above theorem consider the case when both of the sets $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ are the normalized Legendre[10] polynomials $\{(2i+1/2)^{1/2}P_i(x)\}$. The set of first derivatives of these functions are orthogonal[11] with respect to the weight function $1-x^2$

[10] See, for example, D. Jackson, *Fourier series and orthogonal polynomials*, The Carus Monograph no. 6, pp. 45–68.

[11] See Jackson op. cit. Footnote 10 p. 149.

on the interval $(-1, 1)$ so that the hypothesis on $\{\phi_i'(x)\}$ holds in accord with the above theorem. Since

$$N_n^2 = \int_{-1}^{+1} (1-x^2)[P_n'(x)]^2 dx = \frac{2n+1}{2} \cdot \frac{2(n+1)!}{(n-1)!} \cdot \frac{1}{2n+1} = n(n+1)$$

we have[12]

$$\left(\int_{-1}^{+1} (1-x^2)[s_n'(x)]^2 dx\right)^{1/2} \leqq (n(n+1))^{1/2} \left(\int_{-1}^{+1} s_n^2(x) dx\right)^{1/2}$$

$$= O(n)\left(\int_{-1}^{+1} s_n^2(x) dx\right)^{1/2}.$$

Similar results can be obtained with other Jacobi, the Laguerre, and the Hermite polynomials.[13] In the last instance we take $\{\phi_i(x)\} = \{H_i(x)/(2\pi)^{1/4}(i!)^{1/2}\}$ and $\{\psi_i(x)\} = \{e^{-x^2/4} H_i(x)/(2\pi)^{1/4}(i!)^{1/2}\}$. Since $H_i'(x) = iH_{i-1}(x)$, then $\sigma(x) = e^{-x^2/2}$ makes the set of functions $\{\sigma^{1/2}(x)\phi_i'(x)\}$ orthogonal on $(-\infty, +\infty)$. The normalization factor $N_n = (\int_a^b \sigma(x)[\phi_n'(x)]^2 dx)^{1/2} = n^{1/2}$ so that

$$\left(\int_{-\infty}^{+\infty} e^{-x^2/2}[s_n'(x)]^2 dx\right)^{1/2} \leqq n^{1/2}\left(\int_{-\infty}^{+\infty} e^{-x^2/2}[s_n(x)]^2 dx\right)^{1/2}.$$

Another example is of interest in that it gives an application of the above theorem to functions which do not involve polynomials. Let $J_0(\mu_i x)$ denote the Bessel functions of zero order where $\{\mu_i\}$ are the roots[14] of the Bessel functions of the first order, $J_1(\mu) = 0$. Let $\phi_i(x) = J_0(\mu_i x)/k_i$ and $\psi_i(x) = x^{1/2}J_0(\mu_i x)/k_i$, where $k_i^2 = \int_0^1 x J_0^2(\mu_i x) dx = J_0^2(\mu_i)/2$. Since $J_0'(\mu_i x) = -\mu_i J_1(\mu_i x)$ it follows that the set of derivatives, $\{\phi_i'(x)\}$, are orthogonal on the interval $(0, 1)$ with respect to the weight function $\sigma(x) = x$. Thus

$$\int_0^1 x\phi_i'(x)\phi_j'(x) dx = (1/k_i k_j)\int_0^1 x J_0'(\mu_i x) J_0'(\mu_j x) dx$$

$$= (\mu_i\mu_j/k_i k_j)\int_0^1 x J_1(\mu_i x) J_1(\mu_j x) dx = 0$$

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[12] It is of interest to compare this result with Bernstein's theorem for polynomials in C space. See D. Jackson, *Bernstein's theorem and trigonometric approximation*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 225–226.

[13] See Jackson op. cit. Footnote 10 p. 225 Problem 5; p. 227 Problem 7; p. 180 Formula 6.

[14] See, for example, Watson, *Theory of Bessel functions*, pp. 477–521, 576–596. We are making use of the well known formulas, $\int_0^1 x J_k^2(\mu x) dx = (1/2)[J_k^2(\mu) + J_{k+1}^2(\mu)] - (k/\mu)[J_k(\mu)J_{k+1}(\mu)]$ and $\int_0^1 x J_k(\mu_i x) J_k(\mu_j x) dx = [J_k(\mu_j)J_k'(\mu_i) - J_k(\mu_i)J_k'(\mu_j)]/(\mu_i^2 - \mu_j^2)$ when $k = 0$ and $k = 1$.

when $i \neq j$. The normalization factor of $x^{1/2}J_1(\mu_n x)$ is $|J_2(\mu_n)|/2^{1/2}$ so that $N_n^2 = \int_0^1 x[\phi_n'(x)]^2 dx = \mu_n^2 J_2^2(\mu_n)/J_0^2(\mu_n)$. But by the well known[15] identity, $(2/x)J_1(x) = J_0(x) + J_2(x)$, it is seen that when μ_n is a root of $J_1(\mu) = 0$ then $J_0(\mu_n) = -J_2(\mu_n)$ so that the right-hand side of the above equation is μ_n^2 which is known[16] to be of the order n^2. Hence if

$$s_n(x) = \sum_{i=0}^{n} a_i \phi_i(x) = \sum_{i=0}^{n} a_i \frac{J_0(\mu_i x)}{k_i},$$

it follows that

$$\left(\int_0^1 x[s_n'(x)]^2 dx \right)^{1/2} \leqq \mu_n \left(\int_0^1 x[s_n(x)]^2 dx \right)^{1/2}$$

$$= O(n) \left(\int_0^1 x[s_n(x)]^2 dx \right)^{1/2}.$$

4. **Degree of convergence of sums of characteristic solutions.** Let $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ be closed infinite sets of orthonormal solutions of a system of integral equations of form (2) in which the kernel $K(x, t)$ is continuous[17] on an interval $a \leqq x \leqq b$, $a \leqq t \leqq b$. Let the corresponding characteristic numbers be positive and arranged as a non-decreasing sequence with respect to the subscript i. Let $f(x)$ be a continuous function on $a \leqq x \leqq b$ and let

$$\sum_{i=1}^{\infty} \left(\lambda_i^p \int_a^b f(x)\phi_i(x)dx \right)^2$$

be convergent for some value of $p \geqq 1$. Then sufficient hypotheses have been assumed so that the equation of the first kind

$$f(x) = \int_a^b K(x, t)u(t)dt \tag{5}$$

has a solution $u(x)$ in L^2 space.[18]

Let $b_0, b_1, \cdots, b_n$ be arbitrary coefficients and let

$$\tau_{n\lambda}(x) = \frac{b_0\phi_0(x)}{\lambda_0} + \frac{b_1\phi_1(x)}{\lambda_1} + \cdots + \frac{b_n\phi_n(x)}{\lambda_n},$$

$$\bar{\tau}_n(x) = b_0\psi_0(x) + b_1\psi_1(x) + \cdots + b_n\psi_n(x).$$

[15] See Watson op. cit. p. 17. Take $n = 1$.

[16] See Watson op. cit. p. 506. Take $\nu = 0$, $\alpha = 0$.

[17] This hypotheses on $K(x, t)$ can be lightened to include kernels that are summable and of summable square. See Footnote 18.

[18] See E. Picard, *Sur un théorème général relatif aux équations intégrales de première espèce et sur quelques problèmes de physiques mathématique*, Rend. Circ. Math. Palermo vol. 29 (1930) pp. 79–83.

By (2) it follows that $\tau_{n\lambda}(x) = \int_a^b K(x, t)\bar{\tau}_n(t)dt$ so that by subtracting $\tau_{n\lambda}(x)$ from $f(x)$ we have with the aid of Schwarz's inequality,

$$(6) \quad |f(x) - \tau_{n\lambda}(x)| \leqq \left(\int_a^b K^2(x, t)dt\right)^{1/2}\left(\int_a^b [u(t) - \bar{\tau}_n(t)]^2 dt\right)^{1/2}.$$

The above expression shows that any set of the b's that will make $\bar{\tau}_n(t)$ converge in the mean to $u(t)$ will make $\tau_{n\lambda}(x)$ converge uniformly to $f(x)$ as n becomes infinite. In particular, if $b_i = \int_a^b u(t)\psi_i(t)dt$, then

$$\int_a^b [u(t) - \bar{\tau}_n(t)]^2 dt = \sum_{i=n+1}^{\infty}\left(\int_a^b u(t)\psi_i(t)dt\right)^2 = \sum_{i=n+1}^{\infty}\left(\lambda_i\int_a^b f(t)\phi_i(t)dt\right)^2.$$

But this in turn is at most equal to

$$(7) \quad \frac{1}{\lambda_{n+1}^{2p-2}}\sum_{i=n+1}^{\infty}\left(\lambda_i^p\int_a^b f(t)\phi_i(t)dt\right)^2$$

for any $p \geqq 1$ when n is sufficiently large so that $\lambda_{n+1} > 1$.

From (6) and (7) we have the following consequence.

THEOREM C. *If $\{\phi_i(x)\}$ and $\{\psi_i(x)\}$ are closed infinite sets of orthonormal solutions of the system of equations*

$$\phi(x) = \lambda\int_a^b K(x, t)\psi(t)dt, \qquad \psi(x) = \lambda\int_a^b K(t, x)\phi(t)dt$$

corresponding to a set of positive characteristic numbers $\lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n \leqq \cdots$, and if $K(x, t)$ is continuous on $a \leqq x \leqq b$, $a \leqq t \leqq b$, then, if $f(x)$ is continuous on $a \leqq x \leqq b$ and

$$\sum_{i=0}^{\infty}\left(\lambda_i^p\int_a^b f(x)\phi_i(x)dx\right)^2$$

is convergent for some $p \geqq 1$, there exist sums $q_n(x)$ of the set $\{\phi_i(x)\}$ such that

$$|f(x) - q_n(x)| \leqq \frac{I_1\nu_n}{\lambda_n^{p-1}}$$

where I_1 is an absolute constant and $\lim_{n=\infty}\nu_n = 0$.

In connection with Theorem C a further comment is of interest if in (5) we can differentiate $f(x)$ and $\int_a^b K(x, t)dt$ s times, the latter under the integral sign, and if the set of functions $\{\phi^{(s)}(x)\}$ are orthogonal

with respect to a positive weight function $\sigma(x)$ on (a, b). In that case it follows if $f^{(s)}(x)$ is multiplied by $\sigma(x)\phi_i^{(s)}(x)$ and integrated over (a, b),

$$\int_a^b \sigma(x)f^{(s)}(x)\phi_i^{(s)}(x)dx = \int_a^b \left[\int_a^b \sigma(x)K^{(s)}(x, t)\phi_i^{(s)}(x)dx\right]u(t)dt$$

$$= \frac{\overline{N}_i^2}{\lambda_i}\int_a^b \psi_i(t)u(t)dt$$

where $\overline{N}_i$ is the normalizing factor of $\sigma^{1/2}(x)\phi_i^{(s)}(x)$. The series $\sum_{i=0}^{\infty}((\overline{N}_i/\lambda_i)\int_a^b\psi_i(t)u(t)dt)^2$ converges since

$$\sum_{i=0}^{\infty}\left(\int_a^b \sigma^{1/2}(x)f^{(s)}(x)\sigma^{1/2}(x)\,\frac{\phi_i^{(s)}(x)}{\overline{N}_i}\,dx\right)^2$$

is convergent by virtue of Bessel's inequality. But if $\lim_{i=\infty}\overline{N}_i/\lambda_i$ is not zero, then $\sum_{i=0}^{\infty}(\int_a^b\psi_i(t)u(t)dt)^2$ converges and as a consequence $\sum_{i=0}^{\infty}(\lambda_i^p\int_a^b f(x)\phi_i(x)dx)^2$ is convergent for some $p\geqq 1$.

5. **Convergence of the minimizing sums of the integral**[19] $g_n = \int_a^b\rho(x)|f(x)-Q_n(x)|^m dx$, $m>0$ **when** (a, b) **is a finite interval.** Let $f(x)$ be a continuous function on (a, b), and let the weight function $\rho(x)$ be summable and have a positive lower bound v on (a, b). Let $\{\phi_i(x)\}$ be a set of characteristic solutions of conjugate sets of integral equations of form (2). Let these solutions correspond to a set of characteristic numbers $\{\lambda_i\}$ that are in the order $0<\lambda_0\leqq\lambda_1 \leqq \cdots \leqq\lambda_n$. Let $\int_a^b K(x, t)dt$ be differentiable with respect to x under the integral sign and let $\int_a^b[\partial K(x, t)/\partial x]^2dt$ as a function of x exist almost everywhere. Let $Q_n(x)=\sum_{i=0}^n a_i\phi_i(x)$ be any sum of the first $n+1$ functions of the set $\{\phi_i(x)\}$, and let $\Phi_n(x)$ denote any one[20] of these that makes g_n the least. Let ϵ_n be the maximum for $|f(x)-Q_n(x)|$ on (a, b). By following the steps Jackson has taken to investigate convergence of minimizing trigonometric sums and polynomials[21] we are led, by virtue of (3) of Theorem A to the following result.

THEOREM D. *If $Q_n(x)$ is an arbitrary sum of the first $n+1$ functions of the set $\{\phi_i(x)\}$, and $\Phi_n(x)$ is a corresponding minimizing sum of*

[19] For a proof of the existence of minimizing sums see D. Jackson, *On functions of closest approximation*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 117–128. See also vol. 25 (1923) pp. 333–337.

[20] If $m\geqq 1$, the minimizing function is unique. See Jackson op. cit. Footnote 19.

[21] See Jackson op. cit. Footnote 1.

g_n there exists a positive factor L independent of the characteristic numbers $\{\lambda_i\}$ and ϵ_n such that

$$|f(x) - \Phi_n(x)| \leqq L\lambda_n^{1/m}\epsilon_n.$$

Moreover, since L depends upon $(\int_a^b [\partial K(x, t)/\partial x]^2 dt)^{1/2}$, it is bounded on $a \leqq x \leqq b$ except perhaps on a set of points of measure zero.

Thus, since ϵ_n is the maximum of $|f(x) - Q_n(x)|$ on (a, b), the question of convergence of $\Phi_n(x)$ to $f(x)$ on the intervals of $a \leqq x \leqq b$, where L exists, is made to depend upon whether sums $Q_n(x)$ exist for an infinite sequence of values of n such that

$$\lim_{n=\infty} \lambda_n^{1/m}\epsilon_n = 0.$$

This is insured by Theorem C if the series

$$\sum_{i=0}^{\infty}\left(\lambda_i^p \int_a^b f(x)\phi_i(x)dx\right)^2$$

is convergent for some $p \geqq (1/m) + 1$.

Theorem D can be strengthened[22] considerably through the application of Theorem B if in addition to the property of orthonormality on the set $\{\phi_i(x)\}$, the set of derivatives $\{\phi_i'(x)\}$ are orthogonal on (a, b) with respect to a continuous non-vanishing weight function $\sigma(x)$. The essential point is that λ_n is usually large in comparison with the normalizing factor N_n, as for example in the integral equation that was cited earlier (p. 71). Here $\lambda_n = O(n^2)$ while N_n of the set of derivatives of $\{\sin nx\}$, that is $\{n \cos nx\}$, is of order n.

In the subsequent discussion we shall also assume that $\int_a^b |d\sigma^{1/2}(x)/dx|\,dx$ exists. Using the same definitions and notations as in the preceding discussion, we let $\pi_n(x) = \Phi_n(x) - Q_n(x)$ and $r_n(x) = f(x) - Q_n(x)$. Let η be the maximum of $|\pi_n(x)|$ on (a, b) and let x_0 be the abscissa at which this maximum is attained. Since

$$\sigma^{1/2}(x)\pi_n(x) - \sigma^{1/2}(x_0)\pi_n(x_0)$$

$$= \int_{x_0}^{x} \frac{d}{dx}[\sigma^{1/2}(x)\pi_n(x)]dx = \int_{x_0}^{x} \sigma^{1/2}(x)\pi_n'(x)dx + \int_{x_0}^{x} \pi_n(x)\frac{d}{dx}\sigma^{1/2}(x)dx$$

for any x on the interval (a, b), we have with the aid of Schwarz's inequality,

[22] In this section of the paper we can extend our discussion to the case when the interval (a, b) is infinite. The treatment requires further hypotheses on the functions $\sigma(x)$, $\{\phi_i(x)\}$ and $\{\phi_i'(x)\}$. In general the details are similar and are therefore omitted.

$$\begin{aligned}&|\sigma^{1/2}(x)\pi_n(x) - \sigma^{1/2}(x_0)\pi_n(x_0)| \\ &\qquad \leqq |x - x_0|^{1/2}\left(\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|\right)^{1/2} \\ &\qquad\quad + \int_{x_0}^{x}|\pi_n(x)|\left|\frac{d}{dx}\sigma^{1/2}(x)\right||dx|.\end{aligned}$$

But since x_0 and x are on (a, b) by Theorem B it follows that

$$\begin{aligned}\left(\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|\right)^{1/2} &\leqq \left(\int_{a}^{b}\sigma(x)[\pi_n'(x)]^2dx\right)^{1/2} \\ &\leqq N_n\left(\int_{a}^{b}\pi_n^2(x)dx\right)^{1/2}\end{aligned}$$

so that

$$\begin{aligned}&|x - x_0|^{1/2}\left(\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|\right)^{1/2} \\ &\quad + \int_{x_0}^{x}|\pi_n(x)|\left|\frac{d}{dx}\sigma^{1/2}(x)\right||dx| \\ &\qquad \leqq |x - x_0|^{1/2}N_n(b - a)^{1/2}\eta + |x - x_0|\eta H\end{aligned} \tag{8}$$

where H is a constant independent of x and N_n. If N_n is large and

$$|x - x_0| < \frac{\sigma(x_0)}{16N_n^2(b - a)}$$

then

$$\begin{aligned}&|\sigma^{1/2}(x)\pi_n(x) - \sigma^{1/2}(x_0)\pi_n(x_0)| \\ &\qquad \leqq \frac{\eta}{4}\sigma^{1/2}(x_0) + \frac{\eta H\sigma(x_0)}{16N_n^2(b - a)} < \frac{\eta}{2}\sigma^{1/2}(x_0),\end{aligned}$$

whence $|\sigma^{1/2}(x)\pi_n(x)| > (\eta/2)\sigma^{1/2}(x_0)$. If w is the maximum of $\sigma^{1/2}(x)$ on (a, b), then $|\pi_n(x)| > (\eta/2)\sigma^{1/2}(x_0)/w$.

Let γ_n be the minimum of g_n. For a value of x on (a, b) at a distance not greater than $\sigma(x_0)/16N_n^2(b-a)$ from x_0,

$$\gamma_n > v|\pi_n(x) - r_n(x)|^m\frac{\sigma(x_0)}{16N_n^2(b - a)}.$$

Thus, for the case when $(\eta/4)(\sigma^{1/2}(x_0)/w) > \epsilon_n$ we have, since $\gamma_n < I_2\epsilon_n^m$, $I_2 = \int_a^b\rho(x)dx$,

$$(9)\qquad \eta < \frac{4w}{\sigma^{1/2}(x_0)}\left(\frac{16[b-a]}{v}\gamma_n N_n^2\right)^{1/m} < s_1 N_n^{2/m}\epsilon_n$$

where s_1 is an absolute constant.

On the other hand, if $(\eta/4)(\sigma^{1/2}(x_0)/w) < \epsilon_n$, we have

$$\eta < \frac{4w}{\sigma^{1/2}(x_0)}\epsilon_n$$

so that for an N_n sufficiently large, (9) is a universal bound for η.

Hence, from the inequality

$$\begin{aligned}|f(x) - \Phi_n(x)| &\leqq |f(x) - Q_n(x)| + |\Phi_n(x) - Q_n(x)| \\ &= |r_n(x)| + |\pi_n(x)|\end{aligned}$$

we have the following result.

THEOREM E. *If in addition to the hypotheses placed on $\rho(x)$, $f(x)$, and the set $\{\phi_i(x)\}$ in Theorem* D *it is assumed that the set of functions $\{\phi_i'(x)\}$ are orthogonal with respect to a continuous positive weight function $\sigma(x)$, and if $\int_a^b |d\sigma^{1/2}(x)/dx|\,dx$ exists, then there exists a constant s_2 independent of N_n, ϵ_n, and also of x such that*

$$|f(x) - \Phi_n(x)| \leqq s_2 N^{2/m}\epsilon_n.$$

Theorem E is subject to further refinement if certain further hypotheses are assumed on the set $\{\phi_i(x)\}$. If the set of functions $\{\phi_i'(x)\}$ are continuous on the interval (a, b), (the condition of orthogonality only requires that the set $\{\phi_i'(x)\}$ with respect to $\sigma(x)$ be integrable L^2), and if at the maximum of $|\pi_n(x)|$, $d\pi_n(x)/dx = 0$, then by a law of the mean[23] for integrals it is seen that

$$\frac{\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|\,dx|}{|x - x_0|} \leqq \frac{\int_a^b \sigma(x)[\pi_n'(x)]^2 dx}{b-a}$$

when x is sufficiently close to x_0. Thus the first integral on the left-hand side of (8) can be replaced by

$$\frac{|x - x_0|^{1/2}}{(b-a)^{1/2}}\left(\int_a^b \sigma(x)[\pi_n'(x)]^2 dx\right)^{1/2}$$

[23] By the law of the mean, there exist points ξ and c on the intervals $|x, x_0|$ and (a, b), respectively, such that $\sigma(\xi)[\pi_n'(\xi)]^2 = (\int_{x_0}^{x}\sigma(x)[\pi_n'(x)]^2|dx|)/|x - x_0|$ and $\sigma(c)[\pi_n'(c)]^2 = (\int_a^b \sigma(x)[\pi_n'(x)]^2 dx)/(b-a)$. Since $\pi'(x)$ is continuous and $\pi_n'(x_0) = 0$ it follows, since $\sigma(x)$ is positive and bounded on (a, b), that $\sigma(\xi)[\pi_n'(\xi)]^2$ can be made smaller than $\sigma(c)[\pi_n'(c)]^2$ by taking x sufficiently close to x_0.

and this leads to a bound of order $N^{1/m}\epsilon_n$ for $|f(x)-\Phi_n(x)|$ in contrast with $N^{2/m}\epsilon_n$ of Theorem E. As an example, a relative maximum of $|\pi_n(x)|$ is assured if the following boundary conditions[24] are assumed on $\Phi_n(x)$: $\Phi_n(a)=Q_n(a)$, and $\Phi_n(b)=Q_n(b)$. In that case $\pi_n(a)=\pi_n(b)=0$ and as a consequence of the hypotheses on the functions $\{\phi_i'(x)\}$, $\pi_n'(x)=0$ at the maximum of $|\pi_n(x)|$.

The hypothesis on $\sigma(x)$ may be lightened to include weight functions that vanish at some points along the interval. The only restriction required by the proof is that $\sigma(x)$ shall not vanish at the point where $|\pi_n(x)|$ takes on its maximum, and this in turn is assured by appropriate boundary conditions on the minimizing function $\Phi_n(x)$, such as were mentioned in the last paragraph. This last observation is of importance in working out details for the convergence of sums in terms of the Bessel, Legendre, Laguerre, and the Hermite functions, among a great number of others that involve weight functions. The details of these discussions follow the above general outline and are therefore omitted.

THE UNIVERSITY OF IOWA

[24] The placing of a boundary condition on $\Phi_n(x)$ does not affect the existence of minimizing sums. See for example, D. Jackson, *Problems of approximation with integral boundary conditions*, Amer. J. Math. vol. 55 (1933) pp. 153–166.

DOUBLE COSET MATRICES AND GROUP CHARACTERS

J. S. FRAME

1. **Introduction.** The principal theorem of this paper extends to the characters of the irreducible representations of an intransitive group a theorem proved in an earlier paper by the author[1] for the degrees of the irreducible representations of a transitive group. A by-product of the development is the theorem that the sum of the traces of the permutations of any subgroup of a permutation group is not less than the corresponding sum for one of its cosets.

Every finite permutation group G, of order g and degree n, can be written as a group of permutation matrices. The $n\times n$ matrix $R(\gamma)$ which corresponds to the element γ of G can be written as a direct sum of submatrices $R^t(\gamma)$ of n^t dimensions corresponding to the n^t symbols of a transitive constituent C^t of G.[2] Associated with such a transitive constituent C^t is a class of conjugate subgroups, $H_\tau^t=(\gamma_\tau^t)^{-1}H^t\gamma_\tau^t$, each of order h^t, of which H^t shall be the subgroup leaving fixed the first symbol of C^t, and H_τ^t the subgroup leaving fixed the τth symbol. If γ_α is any element of G, then in the set of h^sh^t group elements $H^s\gamma_\alpha H^t$, each element will appear h_α^{st} times, where h_α^{st} is the order of the cross-cut of the subgroups $H_\alpha^s=\gamma_\alpha^{-1}H^s\gamma_\alpha$ and H^t.

Counting each element of the set just once, we define the double coset H_α^{st} by the formula

$$H_\alpha^{st}=H^s\gamma_\alpha H^t/h_\alpha^{st}. \tag{1.1}$$

Any element from a double coset can be chosen as the defining element γ_α. The inverses of the elements of a double coset H_α^{st} themselves form a double coset which we call the inverse double coset and denote by $H_{\alpha'}^{ts}$. The product of two double cosets is a linear combination of double cosets. By considering H_α^{rs} as a sum of h^r/h_α^{rs} left cosets of H^s, and $H_{\beta'}^{st}$ as a sum of $h^t/h_{\beta'}^{st}$ right cosets of H^s, and noting that $H^sH^s=h^sH^s$, it is apparent that each element in the product $H_\alpha^{rs}H_{\beta'}^{st}$ occurs a multiple of h^s times. We define the positive integers $c_{\alpha\beta\eta}^{rst}$ by the formula

$$H_\alpha^{rs}H_{\beta'}^{st}/h^s=\sum_\eta c_{\alpha\beta\eta}^{rst}H_\eta^{rt}. \tag{1.2}$$

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[1] J. S. Frame, *The double cosets of a finite group*, Bull. Amer. Math. Soc. vol. 47 (1941) p. 459.

[2] Throughout this paper the superscripts will refer to the transitive constituents.

Let μ^{st} be the number of double cosets H_α^s for fixed s, t. Then for fixed η the constants $c_{\alpha\beta\eta}^{rst}$ are elements of a μ^{rs}- by μ^{st}-dimensional rectangular matrix $M^s(H_\eta^{st})$, and the transposed matrix is the matrix $M^s(H_\gamma^{tr})$ for the inverse double coset. For $r=t$ these matrices are square. Their sum, with parameters b_η^t as coefficients, defines the important matrix

$$K^{st} = \sum_\eta b_\eta^t M^s(H_\eta^{st}), \tag{1.3}$$

each of whose elements are linear combinations of the parameters b_η^t.

The principal theorem of this paper states that the determinant of the matrix n^tK^{st} splits into factors χ_i which are linear in the parameters b_η^t, that these factors χ_i are essentially the characters of the irreducible representations Γ_i of G, and that the multiplicity m_i of the linear factor χ_i is $\mu_i^s\mu_i^t$, where μ_i^t is the multiplicity with which Γ_i occurs in the reduction of the transitive group of matrices $R^t(\gamma)$, and where $\sum\mu_i^s\mu_i^t=\mu^{st}$. Finally the numerical coefficient which appears is an integer divisible by n^sn^t. Written as a formula, the theorem reads

(1.4) THEOREM.

$$(n^t)^m \left| K^{st} \right| = n^sn^tA^{st}\prod_i (\chi_i)^{m_i},$$

where $m_i=\mu_i^s\mu_i^t$, $m=\sum m_i=\mu^{st}$, $A^{st}=$ *an integer.*

A more precise definition of χ_i is obtained after writing

$$b_\eta^t = \sum_\lambda k_{\eta\lambda}^t a_\lambda, \tag{1.5}$$

where a_λ are new arbitrary parameters and $k_{\eta\lambda}^t$ is the number of elements of the double coset H_η^{tt} which belong to the class of conjugates C_λ. Then $\chi_i=\chi_i(\sum a_\lambda C_\lambda)$ is defined as the character of the ring element $\sum a_\lambda C_\lambda$ in the representation Γ_i. The integer A^{st} enters in the proof as the squared absolute value of the determinant of a certain transformation matrix. It is not necessarily a rational square.

In the special case of this theorem which appeared in the earlier paper already cited,[3] all parameters a_λ except a_1 were zero, s was equal to t, χ_i was the degree n_i, and the matrix K^{st} was a diagonal matrix with elements there denoted by k_t.

In §2 the matrices of an intransitive group are broken up into blocks corresponding to the transitive constituents, and it is proved that the sum of the traces of the matrices of a subgroup is not less

[3] J. S. Frame, loc. cit.

than the corresponding sum for a coset. In §3 a set of "double coset matrices" V_α^{ts}, closely related to the double cosets H_α^{st}, are found to form a basis for the matrices V permutable with a group of permutation matrices. Their products are expressible in terms of the structure constants $c_{\alpha\beta\eta}^{rst}$ already defined in (1.2). The matrix $R(a)$ corresponding to the ring element $a=\sum a_\lambda C_\lambda$ is expressed in terms of the double coset matrices, and used in §4 to build a matrix of traces $\chi(V_\alpha^{ts}V_\beta^{st}R(a))$ which is shown to be precisely the matrix n^tK^{st} defined by (1.3). A unitary change of coordinates in §5 induces a change of basis from the matrices V_α^{ts} to new basis matrices E_η^{ts} by means of a transformation $(\rho_{\alpha\eta}^{st})$ whose determinant enters in the final factoring of the determinant n^tK^{st} from which the group characters are obtained. An example to show the application of the main theorem is given in §6.

2. **Cosets and submatrices.** The n symbols of the intransitive group G will be assumed to be collected together into transitive constituents C^t of degree n^t, and the n right cosets $H^t\gamma_\tau^t$ will be arranged, in an order which preserves this grouping, so as to form the elements of a basis vector $\boldsymbol{b}$ of n rows and one column. Multiplication of $\boldsymbol{b}$ on the right by an element γ of G permutes these cosets among themselves within each transitive constituent, so a permutation matrix $R(\gamma)$ is defined by the equation

$$\boldsymbol{b}\gamma = R(\gamma)\boldsymbol{b}. \tag{2.1}$$

Similarly if $\boldsymbol{b}'$ and $R'(\gamma)$ are the transposes of $\boldsymbol{b}$ and $R(\gamma)$, and if E is the $n\times n$ unit matrix, we have

$$\boldsymbol{b}'(\gamma E) = \boldsymbol{b}'R'(\gamma) = \boldsymbol{b}'R(\gamma^{-1}). \tag{2.2}$$

Let us define the idempotent matrix E^t to be the $n\times n$ matrix which is the unit matrix for the symbols of the constituent C^t and the zero matrix for the other symbols, and let us define a "completely symmetric" $n\times n$ matrix W each of whose n^2 elements is a 1. Then

$$E=\sum_t E^t, \qquad E^tE^t=E^t, \qquad E^tE^s=0 \quad \text{if} \quad t\neq s. \tag{2.3}$$

Let us further define the *block* C^{ts} to be the $n^t\times n^s$ rectangular submatrix of an $n\times n$ matrix, whose rows correspond to the symbols of C^t and whose columns correspond to the symbols of C^s. We see that the matrix

$$W^{ts}=E^tWE^s \tag{2.4}$$

consists of 1's in the block C^{ts} and 0's elsewhere.

The general matrix $R(\gamma)$ of the intransitive permutation group G

may be broken up into transitive components $R^t(\gamma)$ in the diagonal blocks C^{tt}. We define

$$R^t(\gamma) = E^t R(\gamma) = R(\gamma)E^t, \tag{2.5}$$

and we then have

$$R(\gamma) = \sum_t R^t(\gamma). \tag{2.6}$$

The fact that the group G is transitive in the symbols of the constituent C^t shows that the sum of the matrices $R^t(\gamma)$ for all γ in G, which we denote by $R^t(G)$, is a multiple of the matrix W^{tt}. A simple count proves the following relation:

LEMMA.

$$R^t(G) = h^t W^{tt}. \tag{2.7}$$

A theorem on the traces of subgroups and cosets, suggested to the author by G. Polya as having been experimentally verified but not theoretically proved, is a consequence of this lemma.

(2.8) THEOREM. *The sum of the numbers of symbols left fixed by the individual permutations of a subgroup of any permutation group is greater than or equal to the corresponding sum for the permutations of any coset of that subgroup.*

To prove this theorem we note first that each specified sum is the sum of traces of the elements of a subgroup H or of a coset $H\gamma$, and second that if the theorem holds for every transitive constituent of an intransitive group G, then it holds, by addition, for G as well. Focusing attention on a constituent C^t which is transitive for G, we further subdivide it into sets C^{tu} which are transitive for the given subgroup H. To the corresponding matrices $R^{tu}(H)$ we apply the lemma, showing that each is a square block of dimension n^{tu}, each of whose elements is equal to h/n^{tu}, and whose trace is equal to h. Multiplication of the matrix $R^t(H)$ by a matrix $R(\gamma)$ has the effect of permuting the columns and replacing the diagonal elements either by the same positive integer as before, or else by 0. Thus the trace of $R^t(H\gamma)$ is less than or equal to the trace of $R^t(H)$, which was to be proved.

3. **Double coset matrices and structure constants.** To describe the structure of the general n-dimensional matrix V which is permutable with every permutation matrix $R(\gamma)$, of an intransitive permutation group G, we define

$$V^{ts} = E^t V E^s. \tag{3.1}$$

Then the permutability condition

$$VR(\gamma) = R(\gamma)V \tag{3.2}$$

together with (2.5) implies the relations

$$V^{ts}R^s(\gamma) = E^t VR(\gamma)E^s = E^t R(\gamma)VE^s = R^t(\gamma)V^{ts}, \tag{3.3}$$

which may be expressed in words by saying that the rectangular submatrix V^{ts} of V intertwines the matrices $R^s(\gamma)$ and $R^t(\gamma)$. From (3.3) we obtain the equation

$$V^{ts} = R^t(\gamma)V^{ts}R^s(\gamma^{-1}), \tag{3.4}$$

which states that the rectangular matrix V^{ts} is unchanged if its rows are permuted by the matrix $R^t(\gamma)$ while its columns undergo simultaneously the transposed permutation $[R^s(\gamma)]'$. It follows in general that many of the coefficients in the matrix V^{ts} are necessarily equal "by symmetry." If $E^{ts}_{\tau\sigma}$ is defined to be a matrix which is zero except for a 1 in the τth row and the σth column of the block C^{ts}, if H^t, γ^t_τ, H^s, γ^s_σ are defined as in §1, and θ^t and θ^s are elements of H^t and H^s, respectively, then

$$E^{ts}_{\tau\sigma} = R^t(\theta^t\gamma^t_\tau)^{-1}E^{ts}_{11}R^s(\theta^s\gamma^s_\sigma). \tag{3.5}$$

To obtain an invariant matrix we take the sum of all the transforms of (3.5) by all γ in G. We get

$$\begin{aligned}\sum_{\gamma\in G} R^t(\gamma^{-1})E^{ts}_{\tau\sigma}R^s(\gamma) &= \sum_{\gamma} R^t(\theta^t\gamma^t_\tau\gamma)^{-1}E^{ts}_{11}R^s(\theta^s\gamma^s_\sigma\gamma)\\ &= \sum_{\delta} R^t(\delta^{-1})\,E^{ts}_{11}R^s\left(\theta^s\gamma^s_\sigma(\gamma^t_\tau)^{-1}(\theta^t)^{-1}\right)R^s(\delta)\\ &= \sum_{\gamma} R^t(\gamma^{-1})\,E^{ts}_{11}R^s(\gamma_\alpha)R^s(\gamma)\end{aligned} \tag{3.6}$$

where γ_α is any element chosen from the double coset H^{st}_α which contains $\gamma^s_\sigma(\gamma^t_\tau)^{-1}$. We define the double coset matrix V^{ts}_α by the formula

$$V^{ts}_\alpha = (1/h^{st}_\alpha)\sum_{\gamma\in G} R^t(\gamma^{-1})E^{ts}_{11}R^s(\gamma_\alpha)R^s(\gamma), \tag{3.7}$$

so that its coefficients are all 0's and 1's. In this matrix there are h^t/h^{st}_α 1's in each row and h^s/h^{st}_α 1's in each column of the block C^{ts}, and 0's elsewhere. Furthermore, by averaging equation (3.4) for all γ in G, it is readily seen that every V^{ts} is a linear combination of these basis matrices V^{ts}_α, and in particular that

$$\sum_{\alpha} V_{\alpha}^{ts} = W^{ts}. \tag{3.8}$$

From this discussion it also follows that

$$W^{ts}V_{\alpha}^{sr} = W^{tr}h^{r}/h_{\alpha}^{sr}. \tag{3.9}$$

The number of independent matrices V_{α}^{ts} in the block C^{ts} is equal to the number of double cosets H_{α}^{st}, which was denoted by μ^{st}. Since the transposed matrix $(V_{\alpha}^{ts})' = V_{\alpha'}^{st}$ corresponds to the inverse double coset $(H_{\alpha}^{st})^{-1} = H_{\alpha'}^{ts} = H^{t}\gamma_{\alpha}^{-1}H^{s}/h_{\alpha}^{st}$, it follows that $\mu^{st} = \mu^{ts}$.

The product of two double coset matrices $V_{\beta'}^{ts}$ and V_{α}^{sr} is given in terms of the coefficients $c_{\alpha\beta\eta}^{rst}$ of (1.2) as follows:

$$V_{\beta'}^{ts}V_{\alpha}^{sr} = \sum_{\eta} c_{\alpha\beta\eta}^{rst}V_{\eta}^{tr}. \tag{3.10}$$

To prove this we work from the definition (3.7) and apply (1.2)

$$\begin{aligned}
V_{\beta'}^{ts}V_{\alpha}^{sr} &= (1/h_{\alpha}^{rs}h_{\beta}^{st})\sum_{\gamma,\delta} R^{t}(\gamma^{-1})E_{11}^{ts}R^{s}(\gamma_{\beta'})R^{s}(\gamma)R^{s}(\delta^{-1})E_{11}^{sr}R^{r}(\gamma_{\alpha})R^{r}(\delta) \\
&= (1/h_{\alpha}^{rs}h_{\beta}^{st})\sum_{\gamma} R^{t}(\gamma^{-1})E_{11}^{ts}E_{11}^{sr}R^{r}(\gamma_{\alpha}H^{s}\gamma_{\beta'}\gamma) \\
&= (1/h^{r}h^{s}h^{t})\sum_{\gamma} R^{t}(\gamma^{-1})E_{11}^{tr}R^{r}(H_{\alpha}^{rs}H_{\beta'}^{st})R^{r}(\gamma) \\
&= \sum_{\eta} c_{\alpha\beta\eta}^{rst}\sum_{\gamma} R^{t}(\gamma^{-1})E_{11}^{ts}R^{r}(H_{\eta}^{rt}/h^{r}h^{t})R^{r}(\gamma) \\
&= \sum_{\eta} c_{\alpha\beta\eta}^{rst}V_{\eta}^{tr}.
\end{aligned}$$

By summing equation (3.10) first for β and then for α, and applying (3.9) we obtain

$$\begin{aligned}
W^{ts}V_{\alpha}^{sr} &= \sum_{\beta\eta} c_{\alpha\beta\eta}^{rst}V_{\eta}^{tr} = W^{tr}(h^{r}/h_{\alpha}^{sr}), \\
V_{\beta'}^{ts}W^{sr} &= \sum_{\alpha\eta} c_{\alpha\beta\eta}^{rst}V_{\eta}^{tr} = (h^{t}/h_{\beta}^{ts})W^{tr}.
\end{aligned} \tag{3.11}$$

Hence

$$\sum_{\beta} c_{\alpha\beta\eta}^{rst} = h^{r}/h_{\alpha}^{sr}; \qquad \sum_{\alpha} c_{\alpha\beta\eta}^{rst} = h^{t}/h_{\beta}^{st}; \qquad \sum_{\alpha,\beta} c_{\alpha\beta\eta}^{rst} = n^{s}. \tag{3.12}$$

These sums are the sums for a column, or row, or all elements of the matrix $M^{s}(H_{\eta}^{rt})$ defined in §1.

4. **Conjugate classes and the trace matrix.** The matrix $R(C_{\lambda})$, defined to be the sum of the matrices $R(\gamma)$ for all γ in the class of conjugate elements C_{λ}, and likewise the more general matrix $R(a)$,

where $a=\sum a_\lambda C_\lambda$ is a general element from the central of the group ring of G and a_λ are arbitrary parameters, are both matrices permutable with the group, and as such are expressible in terms of the double coset matrices. They are confined to the diagonal blocks C^{tt}, so we have

$$R^t(C_\lambda)=\sum_\delta g^t_{\delta\lambda}V^{tt}_\delta; \qquad R^t(a)=\sum a_\lambda R^t(C_\lambda). \tag{4.1}$$

The coefficient $g^t_{\eta\lambda}$ is the number of elements of C_λ which belong to any particular right coset of H^u_η.

We are now ready to define and evaluate the important μ^{st}-dimensional trace matrix $X^{st}_{\alpha\beta}(a)$:

$$\begin{aligned}
X^{st}_{\alpha\beta}(a) &\equiv \chi(V^{ts}_\alpha V^{st}_\beta R^t(a)) \\
&= \chi\Big(\sum_\eta c^{tst}_{\alpha\beta\eta}V^{tt}_\eta \sum_\lambda a_\lambda R^t(C_\lambda)\Big) \\
&= \sum_\lambda a_\lambda \chi\Big(\sum_{\eta,\delta} c^{tst}_{\alpha\beta\eta}g^t_{\delta\lambda}V^{tt}_\eta V^{tt}_\delta\Big) \\
&= \sum_{\lambda,\eta} a_\lambda c^{tst}_{\alpha\beta\eta}g^t_{\eta\lambda}\chi(V^{tt}_\eta V^{tt}_\eta) \\
&= \sum_{\lambda,\eta} a_\lambda c^{tst}_{\alpha\beta\eta}k^t_{\eta\lambda}n^t \\
&= n^t\sum_\eta c^{tst}_{\alpha\beta\eta}b^t_\eta.
\end{aligned} \tag{4.2}$$

The letter χ is used to denote the trace of a matrix. The new coefficients $k^t_{\eta\lambda}$ and parameters b^t_η are defined as follows:

$$k^t_{\eta\lambda}=g^t_{\eta\lambda}h^t/h^{tt}_\eta; \qquad b^t_\eta=\sum_\lambda a_\lambda k^t_{\eta\lambda}. \tag{4.3}$$

The integer $k^t_{\eta\lambda}$ is the number of elements of C_λ which occur in the whole double coset H^u_η. In terms of the matrix K^{st} of (1.3), we have

$$X^{st}(a)=n^tK^{st}(a)=n^t\sum_\eta b^t_\eta M^s(H^{tt}_\eta). \tag{4.4}$$

In applying the theory to a given group, as in §6, we first compute the matrices M^s for each double coset, form the sum with parameters b^t_η as coefficients, and factor the resulting determinant into factors linear in the b^t_η. One of these factors will be $(b^t_1+b^t_2+\cdots)$, corresponding to the identity representation. We then compute the integers $k^t_{\eta\lambda}$ by studying the group directly, and replace the parameters

b_η^t by appropriate linear combinations of the parameters a_λ. The coefficients of a_λ so obtained in any linear factor, divided by the number g_λ of elements in C_λ, are proportional to the traces $\chi_{i\lambda}$ of the elements of C_λ in the irreducible representation Γ_i corresponding to that factor.

5. **The determinant of the trace matrix.** A unitary matrix U exists which will simultaneously reduce all the matrices $R(\gamma)$, where γ can now be thought of as an arbitrary element of the group ring of G. The matrix U can be chosen to be zero outside the diagonal blocks C^{tt}, in which we define $U^t = E^t U E^t$. It can further be so chosen that all the equivalent irreducible components in the decomposition of $R(\gamma)$ are actually identical, and we shall assume that this is done, and that the irreducible representation Γ_i of degree n_i occurs with multiplicity μ_i^t in $(U^t)^{-1}R^t(\gamma)U^t$. The intersections of the $n_i\mu_i^t$ rows and $n_i\mu_i^s$ columns of the block C^{ts} which belong to the representation Γ_i will be called a representation *district* D_i^{ts}. The district itself divides up into $\mu_i^t\mu_i^s$ squares of n_i dimensions. With each square is associated a unit matrix $E_{i,kl}^{ts}$, $k=1, \cdots, \mu_i^t; l=1, \cdots, \mu_i^s$, conveniently denoted by E_η^{ts}. Every matrix permutable with all the reduced matrices $U^{-1}R(\gamma)U$ is a linear combination of these matrices E_η^{ts}. In particular we have

$$
\begin{aligned}
U^{-1}V_\alpha^{st}U &= (U^s)^{-1}V_\alpha^{st}U^t = \sum_\eta \rho_{\alpha\eta}^{st}E_\eta^{st}, \\
U^{-1}V_{\alpha'}^{ts}U &= \sum_\eta \tilde{\rho}_{\alpha\eta}^{st}E_{\eta'}^{ts}.
\end{aligned} \tag{5.1}
$$

The number of basis matrices E_η^{ts} for fixed t, s, must equal μ^{st}, the number of V_α^{ts}, so we have the relation

$$
\mu^{st} = \sum_i \mu_i^s\mu_i^t. \tag{5.2}
$$

Thus the coefficients $(\rho_{\alpha\eta}^{st})$ defined by (5.1) form a μ^{st}-dimensional square matrix. Its determinant, denoted by P^{st}, is not zero, since both the $U^{-1}V^{ts}U$ and the E^{ts} form a basis for the same set of matrices. That P^{st} is an algebraic integer (not necessarily rational), divisible by $(n^sn^t)^{1/2}$, can be shown as follows.

A partial reduction of the linear group to split off the identity representation Γ_1 in each constituent C^t can be effected by first replacing U^t by a unitary matrix whose (u, v)th element in the block C^{tt} is $\epsilon^{(u-1)(v-1)}/(n^t)^{1/2}$, ($\epsilon$ an n^tth root of unity), and then effecting the reduction in the remaining n^t-1 dimensions. The coefficients $\rho_{\alpha 1}^{st}$ are thus readily computed explicitly:

$$\rho_{\alpha 1}^{st} = (g/h_\alpha^{st})/(n^s n^t)^{1/2} = (h^s h^t)^{1/2}/h_\alpha^{st}; \tag{5.3}$$

$$\sum_\alpha \rho_{\alpha 1}^{st} = \sum_\alpha (n^s h^s/h_\alpha^{st})/(n^s n^t)^{1/2} = (n^s n^t)^{1/2}. \tag{5.4a}$$

Furthermore, since the matrix W^{ts} of (3.8) is of rank 1 and is represented by 0 except in the district D_1^{ts}, we have

$$\sum_\alpha \rho_{\alpha\eta}^{st} = 0, \qquad \eta \neq 1. \tag{5.4b}$$

Adding all rows of $(\rho_{\alpha\eta}^{st})$ to the first, we find that

$$\mathrm{P}^{st} = (n^s n^t)^{1/2} \mathrm{P}_1^{st}, \tag{5.5}$$

where P_1^{st} is the cofactor of ρ_{11}^{st} in P^{st}. Chopping off the first row and column of each, we multiply the conjugate of $(\rho_{\alpha\eta}^{st})$ by the transpose of this same matrix, and obtain a matrix $(A_{\alpha\beta}^{st})$ of $\mu^{st}-1$ dimensions, with determinant A^{st} given by the formula

$$A^{st} = \bar{\mathrm{P}}_1^{st} \mathrm{P}_1^{st} = \bar{\mathrm{P}}^{st} \mathrm{P}^{st}/n^s n^t. \tag{5.6}$$

The element in row $\alpha-1$ and column $\beta-1$ of $(A_{\alpha\beta}^{st})$ is

$$A_{\alpha\beta}^{st} = \sum_{\eta>1} \bar{\rho}_{\alpha\eta}^{st} \rho_{\beta\eta}^{st} = \sum_{\eta>1} \rho_{\alpha'\eta'}^{ts} \rho_{\beta\eta}^{st} = \sum_{i>1} \sum_{k,l} \rho_{\alpha' i,lk}^{ts} \rho_{\beta i,kl}^{st} \tag{5.7}$$

since the subscripts η and η' stand for the three subscripts (i, kl) and (i, lk), respectively. The rule of combination (3.10) which defines the product of $V_{\alpha'}^{ts}$ and V_β^{st} holds likewise for the representatives of these matrices in the districts corresponding to Γ_i, and we have for fixed i

$$\sum_k \rho_{\alpha' i,hk}^{ts} \rho_{\beta i,kl}^{st} = \sum_\delta c_{\beta\alpha\delta}^{tst} \rho_{\delta i,hl}^{tt}. \tag{5.8}$$

The trace $T_{\delta i}^t = \sum_l \rho_{\delta l,ll}^{tt}$ of the μ_i^t-dimensional matrix representing V_δ^{tt} n_i times in the district D_i^{tt} is an algebraic integer. Hence each element of the matrix $(A_{\alpha\beta}^{st})$,

$$A_{\alpha\beta}^{st} = \sum_{\delta, i>1} c_{\beta\alpha\delta}^{tst} T_{\delta i}^t \tag{5.9}$$

is an algebraic integer, and the determinant A^{st} is an integer.

We now use the transformation (5.1) to obtain a new expression for (4.2):

$$\begin{aligned} X_{\alpha\beta}^{st}(a) &= \chi\Big(\sum_{\xi,\eta} \bar{\rho}_{\alpha\xi}^{st} E_{\xi'}^{ts} \rho_{\beta\eta}^{st} E_\eta^{st} (U^t)^{-1} R^t(a) U^t \Big) \\ &= \sum_\eta \rho_{\beta\eta}^{st} \bar{\rho}_{\alpha\eta}^{st} \chi(E^{tt} (U^t)^{-1} R^t(a) U^t) = \sum_\eta \rho_{\beta\eta}^{st} \bar{\rho}_{\alpha\eta}^{st} \chi_\eta(a). \end{aligned} \tag{5.10}$$

Taking determinants of both sides of (5.10) we obtain

$$|X^{st}_{\alpha\beta}(a)| = P^{st}\bar{P}^{st}\prod_{\eta}\chi_\eta(a). \tag{5.11}$$

The subscript η stands for the three subscripts i, kl, but since $\chi_\eta(a)$ is a character depending on i alone, we group together the $m_i=\mu^k_i\mu^s_i$ equal factors $\chi_i(a)$, use (5.6), and write

$$|X^{st}_{\alpha\beta}(a)| = n^s n^t A^{st}\prod_{i}[\chi_i(a)]^{m_i}. \tag{5.12}$$

Reference to (4.4) gives us the final formula for the principal theorem of (1.4).

THEOREM.

$$(n^t)^m|K^{st}(\sum a_\lambda C_\lambda)| = n^s n^t A^{st}\prod_i\left(\sum_\lambda a_\lambda\chi_i(C_\lambda)\right)^{\mu^s_i\mu^t_i}, \tag{5.13}$$

$$m=\sum_i \mu^s_i\mu^t_i=\mu^{st}.$$

6. **An illustrative example.** To illustrate the application of this theorem to the determination of one of the characters of a fairly complicated "simple" group, we choose the simple group G_{25920}, which is an invariant subgroup of index 2 in the group of isomorphisms of the general cubic surface. It can be represented by a group of permutations R^1 on the 45 tritangent planes (or triangles) of the cubic surface, and also by a group of permutations R^2 on the 27 lines of the cubic surface. These two transitive groups, when completely reduced, have besides the identity one common irreducible component, whose character we shall compute by our theorem (5.13). Let H^1 and H^2 be the subgroups of orders $h^1=g/n^{(1)}=576$ and $h^2=960$, which leave fixed, respectively, a particular triangle T and one of its lines L. The common subgroup of order $h^{12}_1=192$ leaves both fixed. There are just two double cosets H^{12}_α, namely $H^{12}_1=H^1H^2/192$, consisting of 2880 elements which transform T into one of the five triangles on L, and a second double coset H^{12}_2 consisting of 23040 elements which transform T into one of the other 40 triangles. Hence there are just two irreducible components common to R^1 and R^2, the identity and one other. There are three double cosets H^{11}_α, namely $H^{11}_1=H^1$ containing 1 coset leaving T fixed, H^{11}_2 containing 12 cosets which take T into one of the 12 "adjacent" triangles having a line in common with it, and H^{11}_3 containing 32 cosets taking T into a non-adjacent triangle.

To determine the constants $c^{tst}_{\alpha\beta\eta}$, we first note that

$$H^{12}_1H^{21}_1/960 = 3H^{11}_1 + H^{11}_2, \tag{6.1}$$

since just 1/5 of the operations of H^1H^2 belong to H^1 and the other 4/5 transform T into an adjacent triangle, and since H_2^{11} contains 12 times as many elements as H_1^{11}. Then we use the formulas (3.12), which become

$$\sum_\alpha c_{\alpha 1\gamma}^{121} = \sum_\alpha c_{1\alpha\gamma}^{121} = h^1/h_1^{12} = 3,$$
$$\sum_\alpha c_{\alpha 2\gamma}^{121} = \sum_\alpha c_{2\alpha\gamma}^{121} = h^1/h_2^{12} = 24. \tag{6.2}$$

The matrices $M^s(H_\gamma^{11})$ may now be written

$$M^2(H_1^{11}) = \begin{pmatrix} 3 & 0 \\ 0 & 24 \end{pmatrix}, \qquad M^2(H_2^{11}) = \begin{pmatrix} 1 & 2 \\ 2 & 22 \end{pmatrix},$$
$$M^2(H_3^{11}) = \begin{pmatrix} 0 & 3 \\ 3 & 21 \end{pmatrix}. \tag{6.3}$$

In factoring the determinant $|K^{21}|$ of (1.3) it will be convenient to omit the superscripts from b_γ^1, $k_{\gamma\lambda}^1$, $g_{\gamma\lambda}^1$, A^{21}. Then

$$|K^{21}| = \begin{vmatrix} 3b_1 + b_2 & 2b_2 + 3b_3 \\ 2b_2 + 3b_3 & 24b_1 + 22b_2 + 21b_3 \end{vmatrix} \tag{6.4}$$
$$= (b_1 + b_2 + b_3)(72b_1 + 18b_2 - 9b_3).$$

The first of these factors corresponds to the identity representation Γ_i. The other factor remains to be investigated. Since $n^1 = 45$, equation (5.13) gives

$$(45)^2(72b_1 + 18b_2 - 9b_3) = 27 \cdot 45A \sum_\lambda a_\lambda \chi(C_\lambda) \tag{6.5}$$

where

$$b_\gamma = \sum_\lambda a_\gamma k_{\gamma\lambda}, \qquad A = P_1^{21}\overline{P}_1^{21}. \tag{6.6}$$

Equating coefficients of a_λ in (6.5) after using (6.6) we have

$$A\chi(C_\lambda) = (5/3)(72k_{1\lambda} + 18k_{2\lambda} - 9k_{3\lambda}). \tag{6.7}$$

The degree d of this representation is $d = 120/A$, which must be an integer less than 27. Noting that $k_{1\lambda} = g_{1\lambda}$, $k_{2\lambda} = 12g_{2\lambda}$, and $k_{3\lambda} = 32g_{3\lambda}$, we may rewrite (6.7) in the form

$$\chi(C_\lambda) = d(g_{1\lambda} + 3g_{2\lambda} - 4g_{3\lambda})$$
$$= g_\lambda \chi_\lambda \tag{6.8}$$

where χ_λ is the trace of a single element of C_λ and g_λ is the number of elements in the class C_λ. To complete the computation it is necessary to find the constants $g_{\pi\lambda}$. Taking the 20 classes of conjugates of the simple group of order 25920 in the order listed in an earlier paper,[4] we calculate $\chi(C_\lambda)$ and χ_λ as follows.

g_λ	$g_{1\lambda}$	$g_{2\lambda}$	$g_{3\lambda}$	$\chi(C_\lambda)$	χ_λ
1	1	0	0	d	$d = 20$
45	13	0	1	$9d$	$d/5 = 4$
270	30	12	3	$54d$	$d/5 = 4$
540	12	12	12	0	$0 = 0$
40	8	0	1	$4d$	$d/10 = 2$
40	8	0	1	$4d$	$d/10 = 2$
360	8	0	11	$-36d$	$-d/10 = -2$
360	8	0	11	$-36d$	$-d/10 = -2$
2160	48	48	48	0	$0 = 0$
2160	48	48	48	0	$0 = 0$
480	32	0	14	$-24d$	$-d/20 = -1$
1440	32	48	26	$72d$	$d/20 = 1$
240	32	12	2	$60d$	$d/4 = 5$
2160	96	60	42	$108d$	$d/20 = 1$
720	64	12	16	$36d$	$d/20 = 1$
720	64	12	16	$36d$	$d/20 = 1$
3240	72	72	72	0	$0 = 0$
5184	0	144	108	0	$0 = 0$
2880	0	48	72	$-144d$	$-d/20 = -1$
2880	0	48	72	$-144d$	$-d/20 = -1$

The scalar product of the columns $\chi(C_\lambda)$ and χ_λ is equal to the order 25920 of the group, so we have $1296d^2/20 = 25920$; $d = 20$; $A = 6$. Hence the degree d is 20, and the last column gives the required character of the irreducible representation of this degree. Eliminating the common components of degree 1 and 20 from the permutations on 45 and 27 symbols, we immediately obtain two other irreducible components of degree 24 and 6, respectively. The components of degree 1 and 20 occur in the Kronecker square of the one of degree 6, and an irreducible component of degree 15 is left. Thus the laborious work of obtaining the character of degree 20 bears fruit in obtaining three more characters with almost no work at all. The theorem of this paper gives a method of digging out some of the characters of a group if simpler methods are not available.

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[4] J. S. Frame. *The simple group of order* 25920, Duke Math. J. vol. 2 (1936) p. 483.

THE ZEROS OF CERTAIN COMPOSITE POLYNOMIALS

MORRIS MARDEN

1. Introduction. If $A_0(z)$ is a given mth degree polynomial and

$$(1.1)\quad A_k(z) = (\beta_k - z)A'_{k-1}(z) + (\gamma_k - k)A_{k-1}(z), \qquad \gamma_k \neq m + k, \quad k = 1, 2, \cdots, n,$$

we may obtain various theorems on the relative location of the zeros of $A_0(z)$ and $A_n(z)$ by the familiar method of first finding such relations for two successive $A_k(z)$ and then iterating the relations n times.

This method has already been employed in the study of the zeros of sequence (1.1) for the following three cases: (1) for all k, $\beta_k = 0$ and γ_k is real;[1] (2) for all k, $\gamma_k = m+1$—a limiting case leading to Grace's theorem,[2] and (3) the limiting case that for all k, as $h \to 0$, $h\beta_k \to \beta_k'$ and $h(\gamma_k - k) \to 1$, in which case $\lim h^k A_k(z)$ is a linear combination of $A_0(z)$ and its first k derivatives.[3]

In the present article we propose to apply the method to the case that *the parameters β_k and γ_k are complex numbers represented by points within certain given regions of the plane.*

To calculate the nth iterate $A_n(z)$ in our case, let us define

$$(1.2)\quad A(z) \equiv A_0(z) \equiv a_0 + a_1 z + \cdots + a_m z^m;$$

$$(1.3)\quad B(z) \equiv (\beta_1 - z)(\beta_2 - z)\cdots(\beta_n - z) \equiv b_0 + b_1 z + \cdots + b_n z^n,$$

$$(1.4)\quad C(z) \equiv (\gamma_1 - 1 - z)(\gamma_2 - 2 - z)\cdots(\gamma_n - n - z);$$

$$S(z, k, p) \equiv B(z) \sum \frac{\gamma_{j_1}^{(k+p)} - 1}{\beta_{j_1} - z} \cdot \frac{\gamma_{j_2}^{(k+p)} - 2}{\beta_{j_2} - z} \cdots \frac{\gamma_{j_{n-p}}^{(k+p)} - (n-p)}{\beta_{j_{n-p}} - z},$$

where $[\gamma_j^{(r)} \equiv \gamma_j - r]$ thus $\gamma_j^{(r)} - j$ is a zero of $C(z+r)$, $p < n$, and the sum is formed for all j_i such that $1 \leqq j_1 < j_2 < \cdots < j_{n-p} \leqq n$;

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[1] See Laguerre, *Oeuvres*, Paris, 1898, vol. 1 pp. 200–202, and G. Polya, *Ueber einem Satz von Laguerre*, Jber. Deutschen Math. Verein. vol. 38 (1929) pp. 161–168.

[2] See Laguerre, *Oeuvres*, vol 1 p. 49, and G. Szegö, *Bemerkungen zu einem Satz von S. H. Grace*, Math. Zeit. vol. 13 (1922) pp. 28–55, p. 33.

[3] See M. Fujiwara, *Eine Bemerkungen uber die elementare Theorie der algebraischen Gleichungen*, Tôhoku Math. J. vol. 9 (1916) pp. 102–108; T. Takagi, *Note on the algebraic equations*, Proceedings of the Physico-Mathematical Society of Japan vol. 3 (1921) pp. 175–179; J. L. Walsh, *On the location of the roots of polynomials*, Bull. Amer. Math. Soc. vol. 30 (1924) p. 52, and M. Marden, *On the zeros of the derivative of a rational function*, Bull. Amer. Math. Soc. vol. 42 (1936) p. 406.

$$S(z, k, n) \equiv B(z) \quad \text{and} \quad S(z, k, p) \equiv 0 \quad \text{for} \quad p > n.$$

Then by repeated use of formula (1.1), we find for

$$D(z) \equiv A_n(z) \equiv d_0 + d_1 z + \cdots + d_m z^m \tag{1.5}$$

the two expressions

$$\begin{aligned} D(z) &= \sum_{p=0}^{n} S(z, 0, p) \frac{d^p A(z)}{dz^p} \\ D(z) &= \sum_{k=0}^{m} \sum_{p=0}^{m-k} \frac{(k+p)!}{k!} S(0, k, p) a_{k+p} z^k. \end{aligned} \tag{1.6}$$

Let us note two special cases of these formulas. First, if $\beta_k = 0$ for all k, then

$$S(0, k, p) = 0 \quad \text{for} \quad p \neq 0, \qquad S(0, k, 0) = C(k)$$

and, hence,

$$D(z) = C(0)a_0 + C(1)a_1 z + \cdots + C(m)a_m z^m. \tag{1.7}$$

Secondly, if, for all k, $\gamma_k = \gamma + 1$, where γ is any constant other than $m, m+1, \cdots, m+n-1$, then

$$\begin{aligned} S(0, k, p) &= (\gamma - k - p)(\gamma - k - p - 1) \cdots (\gamma + 1 - k - n) \sum \beta_{j_1} \beta_{j_2} \cdots \beta_{j_{n-p}} \\ &= (-1)^{n-p}(n-p)! C_{\gamma-k-p, n-p} b_{n-p} \end{aligned}$$

where $C_{r,s} = r(r-1) \cdots (r-s+1)/1 \cdot 2 \cdots s$ and, hence, except for the multiplier $n!$,

$$D(z) = \sum_{k=0}^{m} \sum_{p=0}^{m-k} (-1)^{n-p} C_{n,p}^{-1} C_{\gamma-k-p, n-p} C_{k+p, k} a_{k+p} b_{n-p} z^k \tag{1.8}$$

with $b_{n-p} = 0$ for $p > n$.

In what follows it will be convenient to denote by a script capital $\mathcal{J}$ a region containing all the zeros of a given function $F(z)$. Thus, $\mathcal{A}: |z| \leqq r$ will mean that all the zeros of the polynomial $A(z)$ lie in or on the circle $|z| = r$.

2. **Zeros of two successive $A_k(z)$.** Using the preceding notation, the following lemma may be stated.

LEMMA. *Let $\gamma_j' = \gamma_j - j$ denote the zeros of $C(z)$. Then,*
(a) *$\mathcal{A}_k: r_1 \leqq |z| \leqq r_2$ and $|\beta_k| \leqq \lambda r_1$ imply*

$$\mathcal{A}_{k+1}\colon\ r_1 \min\left[1, \frac{|\gamma_k'| - m\lambda}{|\gamma_k' - m|}\right] \leqq |z| \leqq r_2 \max\left[1, \frac{|\gamma_k'| + m\lambda}{|\gamma_k' - m|}\right]; \tag{2.1}$$

(b) $\mathcal{A}_k\colon |z| \leqq r$ *and* $|\beta_k| \geqq \lambda r$ *imply*

$$\mathcal{A}_{k+1}\colon\ |z| \leqq r \quad \text{and} \quad |z| \geqq r \max\left[1, \frac{m\lambda - |\gamma_k'|}{|m - \gamma_k'|}\right]; \tag{2.2}$$

(c) $\mathcal{A}_k\colon \omega_1 \leqq \arg z \leqq \omega_2$ *with* $\omega_2 - \omega_1 \leqq \pi$ *and* $\beta_k = 0$ *imply*

$$\mathcal{A}_{k+1}\colon\ \omega_1 + \min\left(0, \arg\frac{\gamma_k'}{\gamma_k' - m}\right) \leqq \arg z \leqq \omega_2 + \max\left(0, \arg\frac{\gamma_k}{\gamma_k' - m}\right). \tag{2.3}$$

This lemma may be deduced from the results of a previous paper[4] or may be proved directly as follows.

Let $\mathcal{A}_k$ be a circular region and let ζ be any zero of $A_{k+1}(z)$ outside $\mathcal{A}_k$. Then, by Laguerre's theorem,[5] there exists a point α in $\mathcal{A}_k$ such that $[A_k'(\zeta)/A_k(\zeta)] = m/(\zeta - \alpha)$ and, hence, by (1.1)

$$\zeta = \frac{\gamma_k'\alpha - m\beta_k}{\gamma_k' - m}. \tag{2.4}$$

In particular for $|\beta_k| \leqq \lambda r_1$, if $\mathcal{A}_k\colon |z| \leqq r_2$, then[6] we have that $|\zeta| \leqq r_2(|\gamma_k'| + m\lambda)|\gamma_k' - m|^{-1}$, whereas if $\mathcal{A}_k\colon |z| \geqq r_1$, then $|\zeta| \geqq r_1(|\gamma_k'| - m\lambda)|\gamma_k' - m|^{-1}$. Hence, if all the zeros of $A_k(z)$ lie in the ring $r_1 \leqq |z| \leqq r_2$, an arbitrarily chosen zero of $A_{k+1}(z)$ must lie in the ring (2.1).

If $|\beta_k| \geqq \lambda r$ and $\mathcal{A}_k\colon |z| \leqq r$, then $|\zeta| \geqq r(m\lambda - |\gamma_k'|)|\gamma_k' - m|^{-1}$ and hence the zeros of $A_{k+1}(z)$ not satisfying the first inequality (2.2) must satisfy the second inequality (2.2).

Finally, for $\beta_k = 0$, if $\mathcal{A}_k\colon \omega \leqq \arg z \leqq \omega + \pi$, then $\omega + \arg[\gamma_k'(\gamma_k' - m)^{-1}] \leqq \arg\zeta \leqq \omega + \pi + \arg[\gamma_k'(\gamma_k' - m)^{-1}]$. Setting $\omega = \omega_1$ and $\omega = \omega_2 - \pi$ and combining the results, we conclude that, if all the zeros of $A_k(z)$ lie in the sector $\omega_1 \leqq \arg z \leqq \omega_2$, then all the zeros of $A_{k+1}(z)$ lie in the sector (2.3).

[4] M. Marden, ibid. pp. 400–401. See also J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer. Math. Soc. vol. 24 (1922) p. 169, lemma, and Polya-Szegö, *Aufgaben der Analysis*, Berlin 1925 vol. 2 p. 58, problem 117.

[5] Laguerre, *Oeuvres*, vol. 1 p. 49.

[6] See M. Marden, ibid. p. 402.

3. **Zeros of $A_0(z)$ and $A_n(z)$.** We shall now apply part (1) of the lemma to the successive $A_k(z)$ in order to determine the relative location of the zeros of the polynomials $A(z)\equiv A_0(z)$, $B(z)$, $C(z)$ and $D(z)\equiv A_n(z)$. In addition to the notation used hitherto, we shall use the symbol $\mathfrak{C}(z)$ for the polynomial whose zeros are the moduli of the zeros of $C(z)$:

$$\mathfrak{C}(z) = (|\gamma_1'| - z)(|\gamma_2'| - z)\cdots(|\gamma_n| - z).$$

THEOREM I. *Given the positive constants ρ and λ $(\lambda<1)$. Then,*

(1) $\mathcal{A}$: $|z|\leqq r$, $\mathcal{B}$: $|z|\leqq\lambda r$ *and* $\mathcal{C}$: $\rho|z-m|\geqq|z|+m\lambda$ *imply* $\mathcal{D}$: $|z|\leqq r\max(1,\rho^n)$;

(2) $\mathcal{A}$: $|z|\leqq r$, $\mathcal{B}$: $|z|\leqq\lambda r$ *and* $\mathcal{C}$: $0<\rho|z-m|\leqq|z|+m\lambda$ *with* $\rho\geqq 1$ *imply* $\mathcal{D}$: $|z|\leqq r\left|\mathfrak{C}(-m\lambda)/C(m)\right|$;

(3) $\mathcal{A}$: $|z|\geqq r$, $\mathcal{B}$: $|z|\leqq\lambda r\left|\mathfrak{C}(m\lambda)/C(m)\right|$ *and* $\mathcal{C}$: $\rho|z-m|\geqq|z|-m\lambda>0$ *with* $\rho\leqq 1$ *imply* $\mathcal{D}$: $|z|\geqq r\left|\mathfrak{C}(m\lambda)/C(m)\right|$;

(4) $\mathcal{A}$: $|z|\geqq r$, $\mathcal{B}$: $|z|\leqq\lambda r\min(1,\rho^n)$ *and* $\mathcal{C}$: $0<\rho|z-m|\leqq|z|-m\lambda$ *imply* $\mathcal{D}$: $|z|\geqq r\min(1,\rho^n)$.

To prove this theorem, let us define

$$\mu_k = |m-\gamma_k'|^{-1}(|\gamma_k'|+m\lambda);$$

$$M_k = \max \mu_1^{\sigma_1}\mu_2^{\sigma_2}\cdots\mu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1;$$

$$\nu_k = |m-\gamma_k'|^{-1}(|\gamma_k'|-m\lambda) \quad \text{if} \quad |\gamma_k'|>m\lambda \text{ and}$$

$$\nu_k = 0 \qquad \text{if} \quad |\gamma_k'|\leqq\lambda m;$$

$$N_k = \min \nu_1^{\sigma_1}\nu_2^{\sigma_2}\cdots\nu_k^{\sigma_k}, \qquad \text{where } \sigma_j = 0, 1.$$

If $\mathcal{A}$: $|z|\leqq r$ and $\mathcal{B}$: $|z|\leqq\lambda r$, then by the right side of (2.1)

$$\mathcal{A}_1\text{: } |z|\leqq rM_1,\quad \mathcal{A}_2\text{: } |z|\leqq rM_2,\cdots,\mathcal{A}_n\text{: } |z|\leqq rM_n.$$

Since in part (1) of Theorem I

$$\mu_k\leqq\rho,$$

$M_n=\max(1,\rho^n)$, and, since in part (2) $\mu_k\geqq 1$,

$$M_n = \mu_1\mu_2\cdots\mu_n = \left|\mathfrak{C}(-m\lambda)/C(m)\right|.$$

If $\mathcal{A}$: $|z|\geqq r$ and $\mathcal{B}$: $|z|\leqq\lambda rN_n$, then by the left side of (2.1)

$$\mathcal{A}_1\text{: } |z|\geqq rN_1,\quad \mathcal{A}_2\text{: } |z|\geqq rN_2,\cdots,\mathcal{A}_n\text{: } |z|\geqq rN_n.$$

Since in part (3) of Theorem I $0<\nu_k\leqq\rho\leqq 1$, $N_n=\nu_1\nu_2\cdots\nu_n=\left|\mathfrak{C}(m\lambda)/C(m)\right|$; whereas since in part (4) $\nu_k\geqq\rho$, $N_n=\min(1,\rho^n)$. We have thus established Theorem I.

It is to be noticed that each region $\mathcal{C}$ of Theorem I is bounded by one of the ovals $\rho|m-z| = |z| \pm m\lambda$ of the cartesian curve[7]

$$(3.1)\quad [(\rho^2-1)(x^2+y^2) - 2m\rho^2 x + m^2(\rho^2-\lambda^2)]^2 = 4m^2\lambda^2(x^2+y^2)$$

having ordinary foci at the three points $z=0$, $z=m$ and $z=m(\rho^2-1)^{-1}(\rho^2-\lambda^2)$ and a singular focus at the point $z=m\rho^2(\rho^2-1)^{-1}$. If $\rho>1$, curve (3.1) consists of two nested ovals both enclosing $z=m$ and both excluding $z=0$; in this case, the region $\mathcal{C}$ of part (1) of the theorem is the exterior of the outer oval, $\mathcal{C}$ of part (2) is the interior of the outer oval exclusive of point $z=m$ and $\mathcal{C}$ of part (4) is the interior of the inner oval exclusive of point $z=m$. If $\rho=1$, curve (3.1) degenerates into the hyperbola with foci at $z=0$ and $z=m$ and transverse axis of $m\lambda$; in this case $\mathcal{C}$ of part (1) is the region left of the left branch of the hyperbola, $\mathcal{C}$ of part (2) is the region right of the left branch not including $z=m$, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the left of the right branch and $\mathcal{C}$ of part (4) is the interior of the right branch with point $z=m$ omitted. If $\lambda<\rho<1$, curve (3.1) consists of nested ovals, now however both containing $z=0$ and excluding $z=m$; in this case, $\mathcal{C}$ of part (1) is the interior of the inner oval, $\mathcal{C}$ of part (3) is the region common to the exterior of circle $|z|=m\lambda$ and the interior of the outer oval and $\mathcal{C}$ of part (4) is the exterior of the outer oval exclusive of point $z=m$. In the latter case, if $\rho\to\lambda$, the inner oval shrinks to a point and hence, for $\rho<\lambda$, $\mathcal{C}$ of part (1) is a null-set, and the $\mathcal{C}$'s of parts (3) and (4) are those described for $\lambda<\rho<1$.

In the foregoing discussion, we have implied that $\lambda\neq 0$. If $\lambda=0$, curve (3.1) degenerates into the dipolar circle $\rho|z-m| = |z|$ and $D(z)$ is given by formula (1.7). We may thus state the following corollary.

COROLLARY. *If all the zeros of a polynomial* $A(z)=a_0+a_1z+\cdots+a_mz^m$ *lie in the ring* $0\leqq r_1\leqq|z|\leqq r_2\leqq\infty$ *and if all the zeros of an nth degree polynomial* $C(z)$ *lie in the connected region bounded by the circles* $|z|=\rho_1|z-m|$ *and* $|z|=\rho_2|z-m|$ *with* $\rho_1\leqq\rho_2$, *then all the zeros of the polynomial* $D(z)=C(0)a_0+C(1)a_1z+\cdots+C(m)a_mz^m$ *lie in the ring*[8]

$$(3.2)\qquad r_1 \min\,(1,\rho_1^n) \leqq |z| \leqq r_2 \max\,(1,\rho_2^n).$$

If $\rho_2<1$, *the left side of* (3.2) *may be replaced by the then larger number*

[7] See G. Loria, *Curve piane speciali*, Milan, 1930, vol. I pp. 212–214.

[8] For the cases (1) $r_1=0$, $\rho_1=0$, $\rho_2=1$; (2) $r_2=\infty$, $\rho_1=1$, $\rho_2=\infty$; and (3) $r_1=r_2$, $\rho_1=\rho_2=1$, see N. Obrechkoff, *Sur les zeros des polynômes*, C. R. Acad. Sci. Paris vol. 209 (1939) pp. 1270–1272, and L. Weisner, *Roots of certain classes of polynomials*, Bull. Amer. Math. Soc. vol. 48 (1942) p. 283–286.

$r_1\left|C(0)/C(m)\right|$ and, if $1<\rho_1$, the right side may be replaced by the then smaller number $r_2\left|C(0)/C(m)\right|$.

So far we have applied part (1) of the lemma to the successive $A_k(z)$. Similarly, if we apply part (3) of the lemma and formula (1.7), we may obtain the following result.

THEOREM II. *If all the zeros of the polynomial $A(z)=a_0+a_1z+\cdots+a_mz^m$ are in the sector $\omega_1\leqq\arg z\leqq\omega_2$ with $\omega_2-\omega_1=\omega\leqq\pi$, and if all the zeros of an nth degree polynomial $C(z)$ are in the lune $\theta_1\leqq\arg\left[z/(z-m)\right]\leqq\theta_2$ with $\left|\theta_1\right|+\left|\theta_2\right|\leqq(\pi-\omega)/n$, then all the zeros of the polynomial $D(z)=a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m$ lie in the sector*

$$\omega_1+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2+\max(0,n\theta_2). \tag{3.3}$$

If $\theta_2<0$, $\min(0,n\theta_1)$ may be replaced in (3.3) by the then larger number $\arg C(0)/C(m)$ and, if $0<\theta_1$, $\max(0,n\theta_2)$ may be replaced by the then smaller number $\arg C(0)/C(m)$.

4. **Entire functions.** Theorem II and the corollary to Theorem I may be generalized at once through replacing

$$\begin{aligned}D(z)&=a_0C(0)+a_1C(1)z+\cdots+a_mC(m)z^m\\&=\delta(\delta_1-z)(\delta_2-z)\cdots(\delta_m-z)\end{aligned}$$

by

$$F(z)=a_0E(0)+a_1E(1)z+\cdots+a_mE(m)z^m,$$

where

$$E(z)=e^{\lambda z}C(z)\quad\text{and}\quad\lambda=\mu+i\nu.$$

In fact, since

$$F(z)=\sum_{k=0}^{m}a_kC(k)e^{\lambda kz}z^k=D(e^{\lambda z})=\delta e^{m\lambda}\prod_{k=1}^{m}(\delta_ke^{-\lambda}-z),$$

the substitution of $E(z)$ and $F(z)$ for $C(z)$ and $D(z)$ would require only the following changes: in the corollary to Theorem I, inequality (3.2) becomes

$$e^{-\mu}r_1\min(1,\rho_1^n)\leqq\left|z\right|\leqq e^{-\mu}r_2\max(1,\rho_2^n) \tag{4.1}$$

where $e^{m\mu}\left|E(0)/E(m)\right|$ may replace $\min(1,\rho_1^n)$ if $\rho_2\leqq1$ and $\max(1,\rho_2^n)$ if $\rho_1\leqq1$; in Theorem II, inequality (3.3) becomes

$$\omega_1-\nu+\min(0,n\theta_1)\leqq\arg z\leqq\omega_2-\nu+\max(0,n\theta_2) \tag{4.2}$$

where $[m\nu+\arg E(0)/E(m)]$ may replace $\min(0, n\theta_1)$ if $\theta_2 \leqq 0$ and $\max(0, n\theta_2)$ if $0<\theta_1$.

Furthermore, these results may be extended to entire functions $E(z)$ of genus zero or one provided the zeros of $E(z)$ are assumed to lie in infinite regions, determined by taking $\rho_1=1$ or $\rho_2=1$ and $\theta_1=\theta_2=0$.

THEOREM III. *Given the entire functions*

$$A(z) = \sum_{k=0}^{m} a_k z^k, \qquad E(z) = e^{\lambda_0 z}\prod_{k=1}^{\infty}\left(1-\frac{z}{\gamma_k}\right)e^{\lambda_k z},$$

$$F(z) = \sum_{k=0}^{m} a_k E(k) z^k,$$

where $\lambda_j=\mu_j+i\nu_j$.

(a) *If all the zeros of* $A(z)$ *lie in the ring* $0 \leqq r_1 \leqq |z| \leqq r_2 \leqq \infty$, *if all the zeros of* $E(z)$ *lie in the region* $\rho_1 \leqq |z/(z-m)| \leqq \rho_2$ *with at least one number* ρ_1, ρ_2 *unity, and if* $\mu_0+\mu_1+\cdots\to\mu$, *then all the zeros of* $F(z)$ *lie in the ring* $K_1e^{-\mu}r_1 \leqq |z| \leqq K_2e^{-\mu}r_2$, *where* $K_1=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $\rho_1=1$ *or* $\rho_1<1$ *and* $K_2=1$ *or* $e^{\mu m}|E(0)/E(m)|$ *according as* $1=\rho_2$ *or* $1<\rho_2$.

(b) *If all the zeros of* $A(z)$ *lie in the sector* $\omega_1 \leqq \arg z \leqq \omega_2$ *with* $\omega_2-\omega_1 \leqq \pi$, *if all the zeros of* $E(z)$ *lie on the real axis outside of the segment* $(0, m)$ *and if* $\nu_0+\nu_1+\cdots\to\nu$, *then all the zeros of* $F(z)$ *lie in the sector* $\omega_1-\nu \leqq \arg z \leqq \omega_2-\nu$.

Theorem III(b) is a partial generalization of results due to Laguerre and Polya[1] in the case that both $\nu=0$ and all the zeros of $A(z)$ are real. However, it may also be derived from this special case by use of the theorem quoted in problem 153, p. 65, vol. 2 Polya-Szegö's *Aufgaben der Analysis*. For this fact and its following proof, the author is indebted to the referee, Professor Polya.

We may assume without loss of generality that $\nu=0$. Then, according to the Laguerre-Polya results, $\alpha_k=E(k)$ form a set of multipliers such that, if any polynomial $A(z)=a_0+a_1z+\cdots+a_mz^m$ has only positive (negative) zeros, so has also the polynomial $C(z)=\alpha_0a_0+\alpha_1a_1z+\cdots+\alpha_ma_mz^m$. But such multipliers have also the property that, if all the zeros of $A(z)$ lie in the sector $\omega_1 \leqq \arg z \leqq \omega_2$ with $\omega_2-\omega_1 \leqq \pi$, all the zeros of $C(z)$ also lie in this sector. For, since all the zeros of $(1+z)^m$ are negative, the zeros of polynomial

$$G(z) = \alpha_0 + C_{m,1}\alpha_1 z + C_{m,2}\alpha_2 z^2 + \cdots + \alpha_m z^m$$

are also all negative, and, since the sector is a convex region contain-

ing the origin, the theorem from Polya-Szegö may be applied with the $F(z)$ of the theorem taken as $A(z)$. Theorem III(b) then follows immediately.

As an application of Theorem III, let us consider the polynomial $F(z)=\sum_{k=0}^{m}a_kG(k+p)z^k$ where $p>0$ and $G(z)=\Gamma(z)^{-1}=e^{\gamma z}\prod_{n=1}^{\infty}(1+n^{-1}z)e^{-z/n}$, the reciprocal of the gamma function. Since $\nu=0$ and all the zeros of $G(z+p)$ are negative, any sector $\omega_1\leqq\arg z\leqq\omega_2\leqq\pi-\omega_1$ containing all the zeros of $A(z)$ will also contain all the zeros of $F(z)$. For example, if $A(z)=(z-2)(z+1-i)$, then $F(z)=0.5z^2-(1+i)z-2+2i$, which has the zeros $(3.058+0.514i)$ and $(-1.058+1.486i)$, both thus being in the sector $0\leqq\arg z\leqq 135°$ containing the zeros of $A(z)$.

UNIVERSITY OF WISCONSIN AT MILWAUKEE

ON THE EXTENSION OF A VECTOR FUNCTION SO AS TO PRESERVE A LIPSCHITZ CONDITION

F. A. VALENTINE

1. **Introduction.** Let V be a two-dimensional Euclidean space, and let x be a vector ranging over V. The vector function $f(x)$ is to be a vector in V defined over a set S of the space V. The Euclidean distance between any two points x and y in the plane is denoted by $|x-y|$. Furthermore $f(x)$ is to satisfy a Lipschitz condition, so that there exists a positive constant K such that

$$|f(x_1)-f(x_2)|\leqq K|x_1-x_2| \tag{1}$$

holds for all pairs x_1 and x_2 in S.

In event $f(x)$ is a real-valued function of a variable x ranging over a set S of a metric space, then the extension of the definition of $f(x)$ to any set $T\supset S$ so as to satisfy the condition (1) has been accomplished.[1] The present paper establishes the result that the *vector* function $f(x)$ can be extended to any set $T\supset S$ so as to satisfy the Lipschitz condition with the same constant K. In §3 it is shown how the method used to obtain the above result can be applied to yield an extension for the case considered by McShane.[2] If $f(x)$ has its

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[1] E. J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 837–842.

[2] Loc. cit.

functional values in a metric space then, in general, it cannot be extended so as to preserve (1). The geometric theorems which arise in forming the extension prove to be of as much interest as the extension itself. In the following section it is shown that this extension is a consequence of the following theorem about sets of circles in the plane. A circle with center x_i and radius r_i is the two-dimensional set of points x for which $|x-x_i| \leqq r_i$.

THEOREM 1. *Consider in the plane two sets of circles M and M' such that to each circle in M there corresponds a circle in M'. Let $C_i' \in M'$, having center x_i' and radius r_i', correspond to $C_i \in M$, having center y_i and radius r_i.*

Suppose that

$$r_i' = r_i$$

for all corresponding circles C_i and C_i'. Furthermore suppose that

$$|x_i' - x_j'| \leqq |y_i - y_j| \tag{2}$$

for all corresponding pairs of circles (C_i, C_j) and (C_i', C_j'). Finally suppose that the intersection of all the circles in M is not empty, so that the product

$$\prod_M C_i \neq 0. \tag{3}$$

Then it is true that the intersection of all the circles in M' is such that

$$\prod_{M'} C_i' \neq 0. \tag{4}$$

2. **The extension.** If the set S consists of a single point x_1, the extension of $f(x)$ to a second point x_2 is trivial. We will let S' be the set in the f-plane which corresponds to the set S in the x-plane.

Suppose that the set S consists of two points x_1 and x_2, and that we wish to extend $f(x)$ to a third point x_3. With the points Kx_j as centers $(j=1, 2)$ draw circles C_j with radii $r_j = K|x_j - x_3|$, respectively. Define x_j' to be

$$x_j' \equiv f(x_j), \qquad j = 1, 2. \tag{5}$$

With x_j' as centers draw circles C_j' with radii $r_j' = r_j$, respectively. Since by hypothesis $|x_1' - x_2'| \leqq K|x_1 - x_2|$, and since by construction the product $C_1 \cdot C_2 \neq 0$, it is true that $C_1' \cdot C_2' \neq 0$. To extend $f(x)$ to x_3 so as to satisfy (1), choose $x_3' \in C_1' \cdot C_2'$. Since $|x_j' - x_3'| \leqq r_j' = r_j = K|x_j - x_3|$, letting $f(x_3) \equiv x_3'$, the function $f(x)$ has been extended from $S = (x_1, x_2)$ to $T = (x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition (1).

The crucial extension, as we shall see, is from $S=(x_1, x_2, x_3)$ to $T=(x_1, x_2, x_3, x_4)$. In order to construct this extension, the following lemma and theorem are essential. For convenience denote the triangular set determined by three points y_1, y_2, y_3 by $\Delta(y_1, y_2, y_3)$.

LEMMA 1. *If three circles C_i' $(i=1, 2, 3)$ with centers x_i', respectively, are such that*

$$C_i' \cdot C_j' \cdot \Delta(x_1', x_2', x_3') \neq 0, \qquad i, j = 1, 2, 3, \tag{6}$$

$$C_1' \cdot C_2' \cdot C_3' \cdot \Delta(x_1', x_2', x_3') = 0, \tag{7}$$

then it is true that[3]

$$B(C_i') \cdot B(C_j') \cdot \Delta(x_1', x_2', x_3') \neq 0, \qquad i, j = 1, 2, 3, \tag{8}$$

where $B(C_i')$ is the boundary of the circle C_i'.

To prove Lemma 1 select three points $x_{ij} \in C_i' \cdot C_j' \cdot \Delta$ $(i=1, 2; j=2, 3; j>i)$ which exist by virtue of (6). Since $\Delta(x_{12}', x_{13}', x_{23}') \subset \Delta(x_1', x_2', x_3')$, in proving (8) it is sufficient to prove

$$B(C_i') \cdot B(C_j') \cdot \Delta(x_{12}', x_{13}', x_{23}') \neq 0. \tag{9}$$

Denote the line segment joining points x and y by $\overline{xy}$. Since C_1' is convex, $\overline{x_{12}'x_{13}'} \subset C_1'$. Also (7) implies that the point x_{23}' is not contained in C_1'. Hence the circle C_1' contains a maximum subsegment $\overline{x_{12}'P_2}$ of $\overline{x_{12}'x_{23}'}$ and a maximum subsegment $\overline{x_{13}'P_3}$ of $\overline{x_{13}'x_{23}'}$, where the points $P_2 \neq P_3 \neq x_{23}'$. Since $\overline{P_2P_3} \subset C_1'$, since $x_{23}' \notin C_1'$, and since $B(C_1')$ cannot intersect the $B[\Delta(P_2, P_3, x_{23}')]$ except at P_2 and P_3, a simple arc of $B(C_1')$ joins P_2 and P_3 and is contained in $\Delta(P_2, P_3, x_{23}')$ and hence in $\Delta(x_{12}', x_{13}', x_{23}')$.

In exactly the same way, there exist points $Q_3 \in \overline{x_{13}'x_{23}'}$, $Q_1 \in \overline{x_{12}'x_{13}'}$, with $Q_1 \neq Q_3 \neq x_{13}'$, such that Q_1 and Q_3 are joined by a simple arc of $B(C_2')$ contained in $\Delta(x_{12}', x_{13}', x_{23}')$. *Suppose* the point P_3 lay between x_{23}' and Q_3 on $\overline{x_{23}'x_{13}'}$. This supposition, together with the facts $C_2' \supset \overline{x_{23}'Q_3}$, $C_1' \supset \overline{x_{13}'P_3}$, implies that $\overline{x_{23}'x_{13}'} \subset C_1' + C_2'$. But since $\overline{x_{23}'x_{13}'} \subset C_3'$, the previous statement implies that $C_1' \cdot C_2' \cdot C_3' \cdot \Delta \neq 0$, contrary to assumption (7). Hence P_3 is *not* between x_{23}' and Q_3 on

[3] Lemma 1 is stated for circles in order to shorten the proof of Theorem 2; however the lemma can be generalized as follows: *Suppose that each of the sets C_1, C_2, C_3, Δ and each of the products $C_i \cdot \Delta$, $C_i \cdot C_j \cdot \Delta$ $(i, j=1, 2, 3)$ is a simply connected plane set having a simple closed curve as its boundary. Furthermore suppose $C_1 \cdot C_2 \cdot C_3 = 0$, $C_i \cdot C_j \cdot B(\Delta) \neq 0$, where $B(\Delta)$ is the boundary of Δ. Then it is true that $B(C_i) \cdot B(C_j) \cdot \Delta \neq 0$.*

$\overline{x'_{23}x'_{13}}$, so that $Q_3 \in \overline{x'_{23}x'_{13}} - \overline{x'_{13}P_3}$. This implies that the points P_2, Q_3, P_3, Q_1 lie on the boundary of $\Delta(x'_{12}, x'_{13}, x'_{23})$ in a clockwise order *in the order* just given.

Since P_2 and P_3 are joined by a simple arc of $B(C'_1)$ contained in $\Delta(x'_{12}, x'_{13}, x'_{23})$, and since Q_1 and Q_3 are joined by an arc of $B(C'_2)$ having the same properties, condition (9) holds with $i=1$, $j=2$. Since we can choose any pair of the circles C'_i and repeat the above argument, condition (9) has been proved, and hence Lemma 1 is established.

THEOREM 2. *Suppose the two triangles* $\Delta(y_1, y_2, y_3)$ *and* $\Delta(x'_1, x'_2, x'_3)$ *are such that*

$$|x'_i - x'_j| \leqq |y_i - y_j|, \qquad i, j = 1, 2, 3. \tag{10}$$

Then to any point y_4 *in the plane there corresponds a point* x'_4 *contained in* $\Delta(x'_1, x'_2, x'_3)$ *such that*

$$|x'_4 - x'_i| \leqq |y_4 - y_i|, \qquad i = 1, 2, 3. \tag{11}$$

To prove Theorem 2, with y_i as centers draw circles C_i with radii $r_i \equiv |y_4 - y_i|$, respectively. Similarly with x'_i as centers draw circles C'_i with radii $r'_i = r_i$, respectively. Since $y_4 \in C_i$, the products $C_i \cdot C_j \cdot \Delta(y_1, y_2, y_3) \neq 0$. Hence condition (10), together with the fact $r'_i = r_i$, implies condition (6).

We first prove that condition (10) implies $C'_1 \cdot C'_2 \cdot C'_3 \cdot \Delta(x'_1, x'_2, x'_3) \neq 0$. *Suppose this were not so.* Let P_{ij} be a point of intersection of $B(C'_i)$ and $B(C'_j)$. Since we suppose (7) holds, Lemma 1 implies that P_{ij} can be chosen so that $P_{ij} \in \Delta(x'_1, x'_2, x'_3)$ $(i=1, 2; j=2, 3; j>i)$. Since we supposed $C'_1 \cdot C'_2 \cdot C'_3 \cdot \Delta = 0$, we have

$$|x'_2 - P_{12}| < |x'_2 - P_{13}|. \tag{12}$$

Denote the angle less than or equal to π, determined by three points a_1, a_2, a_3, with vertex at a_2, by $\angle a_1 a_2 a_3$. Since by definition $|x'_1 - P_{12}| = |x'_1 - P_{13}| = r_1$, condition (12) implies by the law of cosines that

$$\angle\, x'_2 x'_1 P_{12} < \angle\, x'_2 x'_1 P_{13}. \tag{13}$$

Condition (13), together with the fact $P_{13} \in \Delta(x'_1, x'_2, x'_3)$, implies the first of the inequalities

$$\begin{aligned} &\angle\, x'_2 x'_1 P_{12} + \angle\, x'_3 x'_1 P_{13} < \angle\, x'_2 x'_1 x'_3, \\ &\angle\, x'_1 x'_3 P_{13} + \angle\, x'_2 x'_3 P_{23} < \angle\, x'_1 x'_3 x'_2, \\ &\angle\, x'_3 x'_2 P_{23} + \angle\, x'_1 x'_2 P_{12} < \angle\, x'_3 x'_2 x'_1. \end{aligned} \tag{14}$$

The last two inequalities are proved exactly as the first.

The sum of all the angles in the three triangles $\Delta(x_i', P_{ij}, x_j')$ $(i=1, 2; j=2, 3; j>i)$ is 3π. By adding the inequalities in (14), we see that in these three triangles $\Delta(x_i', P_{ij}, x_j')$ the *sum* of those six angles, each of which has its vertex at one of the points x_1', x_2', x_3' is less than π. Hence the sum of the remaining three angles in $\Delta(x_i', P_{ij}, x_j')$ satisfies the condition

$$\sum_{i,j} \angle\, x_i' P_{ij} x_j' > 2\pi, \qquad i = 1, 2; j = 2, 3; j > i. \tag{15}$$

Furthermore for the $\Delta(y_1, y_2, y_3)$ of Theorem 2 we have

$$\sum_{i,j} \angle\, y_i y_4 y_j \leqq 2\pi, \qquad i = 1, 2; j = 2, 3; j > i, \tag{16}$$

the equality holding if $y_4 \in \Delta(y_1, y_2, y_3)$; otherwise the inequality holds. Consequently *at least one* of the three angles in (15) is greater than the corresponding angle in (16). Without loss of generality renumber the angles so that

$$\angle\, x_1' P_{12} x_2' > \angle\, y_1 y_4 y_2. \tag{17}$$

By construction $|x_i' - P_{12}| = |y_i - y_4|$ $(i=1, 2)$. Condition (17) implies by the law of cosines that in the triangles $\Delta(x_1', P_{12}, x_2')$ and $\Delta(y_1', y_4', y_2)$

$$|x_1' - x_2'| > |y_2 - y_1|. \tag{18}$$

This contradicts assumption (10). Hence the assumption that (7) holds is false. Thus choosing a point $x_4' \in C_1' \cdot C_2' \cdot C_3' \cdot \Delta \neq 0$, we have $|x_4' - x_i'| \leqq r_i' = r_i = |y_i - y_4|$ $(i=1, 2, 3)$ and Theorem 2 has been established.

We now readily establish the following theorem.

THEOREM 3. *If the plane vector $f(x)$ is defined on a plane set $S=(x_1, x_2, x_3)$ so as to satisfy the Lipschitz condition* (1), *then it can be extended to any plane set $T=(x_1, x_2, x_3, x_4)$ so as to be Lipschitz preserving. The extension $f(x_4)$ can be defined so as to be contained in the triangle formed by the points $f(x_i)$ $(i=1, 2, 3)$.*

PROOF. In Theorem 2, let

$$y_i \equiv Kx_i, \qquad x_i' \equiv f(x_i), \qquad y_4 \equiv Kx_4, \qquad i = 1, 2, 3.$$

Then the point x_4' in Theorem 2 is the desired extension $f(x_4) \equiv x_4'$, since conditions (10) and (11) imply that

$$|f(x_j) - f(x_k)| \leqq K|x_j - x_k|, \qquad j, k = 1. \cdots, 4.$$

In order to extend $f(x)$ from an arbitrary set S to any set $T \supset S$ *we first prove Theorem* 1. This can be accomplished by means of Theorem 2 and by a theorem of Helly[4] as follows. Choose an arbitrary but fixed set of three circles C_i $(i=1, 2, 3)$ in the set of circles M of Theorem 1. Since in Theorem 1 the hypotheses include the fact that $\prod_M C_i \neq 0$, choose the arbitrary point y_4 in Theorem 2 so that $y_4 \in \prod_{i=1}^{3} C_i$. Then Theorem 2 implies that there exists a point x_4' such that $|x_4' - x_i'| \leqq |y_4 - y_i|$. Since $y_4 \in \prod_{i=1}^{3} C_i$, it is true that $|y_4 - y_i| \leqq r_i$, whence $|x_4' - x_i'| \leqq r_i$. Since $r_i' = r_i$ in Theorem 1, the above inequalities imply that *each set of three circles in* M' *has a point common to all three circles. Now by Helly's theorem, with* $n=2$, *it is true that all the circles in* M' *have a point in common.* Thus Theorem 1 is established.

Now we are in a position to prove the following theorem.

THEOREM 4. *Suppose the plane vector function* $f(x)$ *is defined and satisfies the Lipschitz condition* (1) *on a set* S *of the plane.*

Then if T *is any set containing* S, *it is true that* $f(x)$ *can be extended to* T *so as to preserve the Lipschitz condition* (1). *The extension of* $f(x)$ *can be defined so that the set of points* $(f(x)$, *for* $x \in T)$ *is contained in any prescribed closed convex set containing* S', *where* $S' \equiv (f(x)$, *for* $x \in S)$.

PROOF. First, we prove that if U is any set on which $f(x)$ satisfies the Lipschitz condition (1), and if x_0 is an arbitrary point exterior to U, then $f(x)$ can be extended from U to $U+x_0$ so as to satisfy (1). To do this let x_i be an *arbitrary* point in U, and let

$$x_i' \equiv f(x_i)$$

be the corresponding point in $U' \equiv [f(x),\ x \in U]$. With $y_i = Kx_i$ as center and with radius $r_i \equiv K|x_0 - x_i|$ draw a circle C_i. As x_i ranges over U, denote the set of circles thus defined by M. Similarly with x_i' as center and with radius $r_i' = r_i$ draw a circle C_i', and denote the set of all circles, as x_i' ranges over U', by M'. Since condition (1) holds on U, condition (2) in Theorem 1 holds on U. Furthermore since by construction $r_i = K|x_0 - x_i|$ we have $\prod_M C_i \neq 0$. *Hence Theorem* 1 *implies that* $\prod_{M'} C_i' \neq 0$. Consequently there exists a point x_0', with $x_0' \in \prod_{M'} C_i'$, which implies that

$$|x_0' - x_i'| \leqq r_i \equiv K|x_0 - x_i|$$

[4] E. Helly, Jber. Deutschen Math. Verein vol. 32 (1923) pp. 175–176. The theorem states: *If each* $n+1$ *sets of a family of closed, bounded, convex sets of the* n*-dimensional Euclidean space intersect, then there is a point common to all the sets.*

for all corresponding pairs x_i and x_i' in U and U', respectively. Hence letting $f(x_0) \equiv x_0'$, we have extended $f(x)$ from U to $U+x_0$ so that the Lipschitz condition (1) is preserved. *For convenience denote the above extension from an arbitrary set U to the set $U+x_0$, where x_0 is arbitrary, as extension* E.

The extension from a set S to a region $T \supset S$ of Theorem 4 can be accomplished as follows. Let R be the set of all the points in the plane which have rational coordinates. Extension E of the preceding paragraph implies by *ordinary induction* that $f(x)$ can be extended from the set S to the set $S+R$ so as to satisfy the Lipschitz condition (1). Let $\bar{x}$ be any other point in the plane. In order to effect the extension to $\bar{x}$ let x_i be any sequence of points with rational coordinates such that $\lim_{i=\infty} x_i = \bar{x}$. If we define $U \equiv S + \sum x_i$, $x_0 \equiv \bar{x}$, then extension E of the preceding paragraph yields an extension $f(\bar{x})$ so that $f(x)$ satisfies (1) on $U+x_0$. Condition (1) implies that $\lim_{i=\infty} f(x_i) = f(\bar{x})$. The function $f(x)$ is defined *uniquely* at $\bar{x}$, for let y_i be any other sequence of rational points such that $\lim_{i=\infty} y_i = \bar{x}$. Letting $U \equiv S + \sum y_i$, and $x_0 \equiv \bar{x}$, extension E yields an extension $f^*(\bar{x})$ such that $f(x)$ satisfies (1) on $U+x_0$. Since

$$|f(y_i) - f(x_i)| \leqq K|y_i - x_i|,$$

and since $\lim_{i=\infty} f(x_i) = f(\bar{x})$, it follows that

$$\lim_{i=\infty} f(y_i) = \lim_{i=\infty} f(x_i) = f^*(\bar{x}) = f(\bar{x}). \tag{19}$$

The function $f(x)$ thus defined satisfies the Lipschitz condition (1) on the whole plane. To prove this statement, let x and y be any two points in the plane, and let x_i and y_i be two sequences of rational points such that $\lim_{i=\infty} x_i = x$, $\lim_{i=\infty} y_i = y$. Then the inequalities

$$\begin{aligned} |f(x) - f(y)| &\leqq |f(x) - f(x_i)| + |f(x_i) - f(y_i)| + |f(y_i) - f(y)| \\ &\leqq K|x - x_i| + K|x_i - y_i| + K|y_i - y| \end{aligned}$$

imply, by passage to the limit, that

$$|f(x) - f(y)| \leqq K|x - y|.$$

Thus the first part of Theorem 4 is proved since if we can extend $f(x)$ to the whole plane we can surely extend it to $T \supset S$.

To prove the last sentence of Theorem 4, let L be any closed convex set such that $L \supset S'$. Consider the sets of circles M and M' defined in the first paragraph following Theorem 4, however with $U = S$ and $U' = S'$. Consider *any* two circles C_1' and C_2' in M'. Since the straight line joining the centers of these circles is in L, and since (1) implies

that $C_1' \cdot C_2' \neq 0$, we have $C_1' \cdot C_2' \cdot L \neq 0$. Furthermore Theorem 1 implies that $\prod_{M'} C_i' \neq 0$. Hence the theorem of Helly[5] implies that the product

$$L \cdot \prod_{M'} C_i' \neq 0.$$

Hence in extension E one can choose $x_0' \in L$, whence $S' + x_0' \subset L$. Now by ordinary induction the function $f(x)$ can be extended to the set $S+R$ (where R is the set of points with rational coordinates), so that the set $S'+R' \equiv [f(x), x$ ranges over $S+R] \subset L$. Now the extension of $f(x)$ to the whole plane so that $f(x) \subset L$, x arbitrary, can be constructed. For consider the point $\bar{x}$ in equation (19), where $x_i \subset R$. Since by (19) $\lim_{i=\infty} f(x_i) = f(\bar{x})$, since $f(x_i) \in L$, and since L is closed, it follows that $f(\bar{x}) \in L$. Thus the proof of Theorem 4 has been completed.

3. **Generalizations.** The functions studied by McShane[6] can be easily extended by the method of the preceding section. Let the *one-dimensional real-valued* function $f(x)$ be defined over a set S of a metric space. Furthermore suppose there exists a real-valued function $w(t) \geqq 0$ defined for $0 \leqq t$, and satisfying the conditions

$$w(a) + w(b) \geqq w(a + b), \tag{20}$$

$$w(b) \geqq w(a), \qquad \text{for } b \geqq a. \tag{21}$$

If the function $f(x)$ satisfies the condition

$$|f(x_1) - f(x_2)| \leqq w(\|x_1, x_2\|) \tag{22}$$

for all pairs (x_1, x_2) in S, then $f(x)$ can be extended to the entire space so as to preserve condition (22).

It should be noted that condition (20) does *not* imply that $w(t)$ be concave downward; however, any function $w(t) \geqq 0$, $t \geqq 0$, which is concave downward must satisfy condition (20). The function $w(t)$ is less restricted than that used by McShane.[7]

To prove the above theorem, let $S = (x_1, x_2)$ and suppose x_3 is a third point to which we wish to extend $f(x)$. With $f(x_i)$ $(i = 1, 2)$ as centers draw circles C_i' with radii $r_i' \equiv w(\|x_i, x_3\|)$, respectively. These circles are linear intervals. Suppose $C_1' \cdot C_2' = 0$. Then by (20) and (21) we would have

$$\begin{aligned} |f(x_1) - f(x_2)| &> w(\|x_1, x_3\|) + w(\|x_2, x_3\|) \\ &\geqq w(\|x_1, x_3\| + \|x_3, x_2\|) \geqq w(\|x_1, x_2\|), \end{aligned}$$

[5] Loc. cit.

[6] Loc. cit.

[7] Ibid.

which contradicts the hypothesis $|f(x_1)-f(x_2)| \leqq w(\|x_1, x_2\|)$. Thus $C_1' \cdot C_2' \neq 0$, and any point $x_3' \in C_1' \cdot C_2'$ serves as an extension $f(x_3)$. In order to extend $f(x)$ from an arbitrary set U on which (22) holds, to a set $U+x_0$, with $f(x_i)$ as centers, $x_i \in U$, draw circles C_i' with radii $r_i' \equiv w(\|x_i, x_0\|)$, respectively. Since each *pair* of the circles C_i' intersect, *Helly's*[8] *theorem, with* $n=1$, implies that $\prod C_i' \neq 0$. This implies that $f(x)$ can be extended from U to $U+x_0$ so as to preserve condition (22). The extension of $f(x)$ from S to the whole space now follows by transfinite induction.

An advantage of the above procedure arises from the fact that *if* the extension were impossible, the method would reveal it. Counter-examples exist which show that if $f(x)$ has its values in a metric space, then, in general, $f(x)$ cannot be extended so as to preserve conditions (1) and (22). For example, let x range over a two-dimensional metric space in which all the "unit spheres" are congruent squares having corresponding sides *parallel*. Let $f(x)$ have its values in the two-dimensional Euclidean space. *In Theorem* 1, let C_i $(i=1, 2, 3)$ consist of three unit squares such that $C_1 \cdot C_2 \cdot C_3 \equiv p$ consists of one and only one vertex from each square. Also choose C_i' so that $r_i' = r_i = 1$, $x_1' = (1, 0)$, $x_2' = (-1, 0)$, $x_3' = (0, 1+\epsilon)$, $\epsilon > 0$. The constant ϵ can be chosen so that condition (2) holds. Although condition (3) holds, conclusion (4) fails. Hence one cannot extend $f(x)$ from $S = (x_1, x_2, x_3)$ to $T = (x_1, x_2, x_3, p)$ so as to preserve condition (1).

The first generalization of the material in §§1 and 2 would be to the n-dimensional case, and thence to a Hilbert space. In the n-dimensional case the crux of this generalization lies in the generalization of Theorem 1, since that of Helly applies to the n-dimensional case. In the case of a Hilbert space one would also need a generalization of the theorem of Helly. These matters are still open questions. The author wishes to express his appreciation to Professor Max Zorn and to Dr. W. T. Puckett with whom he has had stimulating conversation concerning these topics.

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[8] Loc. cit.

A FAMILY OF FUNCTIONS AND ITS THEORY OF CONTACT[1]

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Introduction. If $p_1, \cdots, p_n$ are fixed positive integers and $a_1, \cdots, a_n$ arbitrary constants, it is possible so to choose the a_i as to make the function

$$y(x) = \prod_{i=1}^{n} (x - a_i)^{p_i} \tag{1}$$

and its first $p_1 + \cdots + p_n - 1$ derivatives equal to zero for any single value x_0 of x. This is accomplished by taking each a_i equal to x_0. One might say, on this basis, that *the family of polynomials* (1) *has contact of order* $p_1 + \cdots + p_n - 1$, *for every value of* x, *with* $y = 0$.

A more interesting situation is met when we allow the p_i to be any fixed positive numbers, not necessarily integral. In that case $y(x)$ may be a function of many branches, with the quotient of any two branches equal to a constant of modulus unity. For our purposes it suffices to consider the value zero of x. If no a_i is zero, each branch of $y(x)$ will be analytic at $x = 0$, with an expansion

$$c_0 + c_1 x + \cdots + c_s x^s + \cdots$$

where the c_j depend on the a_i. The question which we examine is: *What is the greatest value of* s *such that, by suitably varying the* a_i, *the coefficients* $c_0, \cdots, c_s$ *can be made to approach zero simultaneously*? Such a greatest value of s exists, and will be called, below, *the order of contact of the family* (1) *with* $y = 0$. Denoting the greatest value of s by r, we shall prove that

$$r \leqq q + n - 1 \tag{2}$$

where q is the greatest integer less than $p_1 + \cdots + p_n$. When no proper subset of the p_i has an integral sum, the equality sign holds in (2). For $n = 2$, (2) can be an inequality only when p_1 and p_2 are both integers. For $n \geqq 3$, (2) will certainly be an inequality if some integral power of $y(x)$ is a polynomial of degree not exceeding $q + n - 1$; thus the order of contact of the family

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[1] The problem of this note was suggested by the considerations of our paper *On the singular solutions of algebraic differential equations*, Ann. of Math. (2) vol. 37 (1936) p. 552. See also, W. C. Strodt, Trans. Amer. Math. Soc. vol. 45 (1939) p. 276.

$$y(x) = (x - a_1)^{1/2}(x - a_2)^{1/2}(x - a_3)^{1/2}$$

is two rather than three. Whether this describes all exceptional cases for $n \geqq 3$ is not decided here.

1. **The family of functions.** In what follows, the p_i in (1) will be any fixed positive numbers. A few words are necessary to make clear the meaning of the second member of (1) for given a_i. If the a_i are distinct from one another, we may take any simply connected area containing no a_i and form the product, in the area, of any selection of branches of the n functions $(x-a_i)^{p_i}$. The various products obtainable in this way are continuations of one another and are all branches of a single analytic function, which we consider the second member of (1) to represent. If two or more a_i coincide, two distinct products, as just described, need not be branches of the same analytic function. There may thus be more than one, possibly even a countable infinitude of interpretations of the second member of (1); every such analytic function will be accepted into the n-parameter family of functions (1).

Given any function y, as in (1), its values, for any x which is not an a_i, are equal in modulus; the same is true for every derivative of y.

2. **Order of contact.** Let $\mathfrak{F}$ be a family of analytic functions and $f(x)$ a function[2] analytic at a point x_0. There may exist non-negative integers s which have the property that, for every $\epsilon > 0$, a $g(x)$ exists in $\mathfrak{F}$, with a branch analytic at x_0, such that, for this branch of $g(x)$, $g(x)-f(x)$ and its first s derivatives are less than ϵ in modulus at x_0. If such integers s exist, and if the set of them is bounded, we shall represent the greatest of them by r and shall say that $\mathfrak{F}$ has *contact of order r* with $f(x)$ at x_0. If the s are unbounded, we shall say that $\mathfrak{F}$ has contact of infinite order with $f(x)$ at x_0.

3. **The bound.** We examine now the functions (1). It is apparent that this family has contact of some order with $y=0$ at every point. Indeed, because the family is invariant under the addition of any constant to x, the contact with $y=0$ is the same for all values of x.

Let q be the greatest integer less than $p_1 + \cdots + p_n$. The order of contact of the family (1) with $y=0$ is not less than q. This is seen by taking all a_i equal to zero. We prove the theorem:

THEOREM. *The order of contact which the family* (1) *has with* $y=0$, *for every* x, *does not exceed* $q+n-1$.

---

[2] Not necessarily in $\mathfrak{F}$.

The theorem is readily seen to be true for $n=1$; we employ induction with respect to n. We examine the theorem for $n=r>1$, assuming that it has been established for every n less than r.

We suppose the theorem false for $n=r$. Then, for $n=r$, and for certain positive numbers $p_1, \cdots, p_r$ which stay fixed during our proof, the family (1) has contact with $y=0$, for $x=0$, of order greater than $q+r-1$. Thus, if we denote the jth derivative of y by y_j,[3] we can, for every $\epsilon>0$, fix the a_i in (1) at values distinct from 0 so as to have[4]

$$|y_i(0)| < \epsilon, \qquad i = 0, 1, \cdots, q+r. \tag{3}$$

Let us show that, if ϵ is sufficiently small, each a_i, as just fixed, will have a modulus less than unity. Suppose, for instance, that for some very small ϵ, $|a_1| \geqq 1$. Then $y(x)/(x-a_1)^{p_1}$ will be very small, together with its first $q+r$ derivatives, at $x=0$. This, by the case of $n=r-1$, is impossible.

We now put

$$\alpha(x) = (x-a_1)\cdots(x-a_r);$$

$$\beta(x) = \alpha(x)\left[\frac{p_1}{x-a_1} + \cdots + \frac{p_r}{x-a_r}\right]. \tag{4}$$

We have

$$\alpha(x)y_1 - \beta(x)y = 0. \tag{5}$$

The polynomial β is of degree $r-1$. Its $(r-1)$st derivative is

$$(r-1)!(p_1 + \cdots + p_r). \tag{6}$$

We differentiate (5) $j-1$ times, where $j \geqq 1$. Indicating derivatives of α and β by subscripts, we find that

$$\alpha y_j + [(j-1)\alpha_1 - \beta]y_{j-1} + \left[\frac{(j-1)(j-2)}{2!}\alpha_2 - (j-1)\beta_1\right]y_{j-2} + \cdots - \beta_{j-1}y = 0. \tag{7}$$

For $j \geqq r$, (7) becomes, because of the degrees of α and β,

$$\alpha y_j + \cdots + (j-1)!\left[\frac{\alpha_r}{r!(j-r-1)!} - \frac{\beta_{r-1}}{(r-1)!(j-r)!}\right]y_{j-r} = 0. \tag{8}$$

[3] $y_0 = y$.

[4] If y is analytic at $x=0$ when certain h of the a_i, say $a_1, \cdots, a_h$, are zero while no other a_i vanish, it must be that $p_1 + \cdots + p_h$ is integral. Thus, if $a_1, \cdots, a_h$ are changed to a common value slightly different from zero, y and any specified finite set of its derivatives will undergo only a slight change at $x=0$.

The coefficient of y_{j-r} in (8) is a constant, which, if we have regard to (6) and notice that $\alpha_r = r!$, is seen to be zero if and only if

$$p_1 + \cdots + p_r = j - r. \tag{9}$$

Let p represent $p_1 + \cdots + p_r$. If, in (1), the a_i are all multiplied by a number m, the values of $y_j(0)$ are multiplied by m^{p-j}. If $|m| > 1$, each $y_j(0)$ with $j > q$ will be multiplied by a number of modulus not greater than unity.

We consider a $y(x)$, (with definite a_i), which satisfies (3) for some very small ϵ. Let m be such that the greatest of the quantities $|ma_i|$, $i = 1, \cdots, r$, has unity for modulus. Then, by what follows (3), $|m| > 1$. Let

$$\bar{y}(x) = \prod_{i=1}^{r} (x - ma_i)^{p_i}.$$

We inspect the relation (8) as formed for $\bar{y}$. First we let $j = q + r$. In that case, (9) cannot hold. Every $|\bar{y}_i(0)|$ with $q < i \leqq q + r$ is small. Furthermore, because $|ma_i| \leqq 1$, $i = 1, \cdots, r$, there are bounds, independent of ϵ, for the values of the coefficients in (8) at $x = 0$. We infer that $|\bar{y}_q(0)|$ is small. Now, supposing that $q > 0$, let $j = q + r - 1$. We find from (8) that $|\bar{y}_{q-1}(0)|$ is small. Continuing, we find that every $|\bar{y}_i(0)|$ with $i \leqq q + r$ is small.

Let g be such that $|ma_g| = 1$. Then the function

$$\bar{y}(x)/(x - ma_g)^{p_g} \tag{10}$$

is small, together with its first $q + r$ derivatives, for $x = 0$. It is clear that we can use a single g and obtain a sequence of functions (10) which is such that the values at $x = 0$ of the kth function of the sequence and its first $q + r$ derivatives tend toward zero as k increases. By the case of $n = r - 1$, this is impossible. The theorem is proved.

4. **Attainment of bound.** We prove, for $n > 1$, the theorem:

THEOREM. *If no proper subset of the p_i has an integral sum, the family* (1) *has, for every x, contact with $y = 0$ of order precisely $q + n - 1$.*

It suffices to show that, when the p_i satisfy the hypothesis, there are values of the a_i distinct from zero such that $y_j(0) = 0$, $j = q + 1, \cdots, q + n - 1$. Such a_i being found, we can multiply them by a small m distinct from zero and obtain a function (1) which is small, for $x = 0$, together with its first $q + n - 1$ derivatives.

The existence of a_i as just described will be established if we can prove that there are numbers $b_1, \cdots, b_n$, distinct from zero, such that the function

$$z = \prod_{i=1}^{n} (1 + b_i x)^{p_i}$$

has derivatives, from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. The $n-1$ derivatives in question, which we represent by $Z_{q+1}, \cdots, Z_{q+n-1}$, are homogeneous polynomials in the n letters b_i. When the Z_q are equated to zero, they determine a non-vacuous algebraic manifold each of whose essential irreducible components is of dimension not less than unity.[5] Thus there is at least one set of numbers $b_1, \cdots, b_n$ which annul the Z_j and are not all zero. We assume in what follows that there is such a set in which the b_i are not all distinct from zero, and prove that some proper subset of the p_i has an integral sum.

We may now work under the assumption that, for some integer t with $0<t<n$, there exist numbers $c_1, \cdots, c_t$, all distinct from zero, such that the function

$$u = \prod_{i=1}^{t} (1 + c_i x)^{p_i}$$

has derivatives from the $(q+1)$st to the $(q+n-1)$st inclusive which vanish for $x=0$. If we put $d_i = -1/c_i$, we find that the function

$$v = \prod_{i=1}^{t} (x - d_i)^{p_i} \tag{11}$$

has derivatives from order $q+1$ through order $q+n-1$ which vanish for $x=0$. For the derivatives v_j of v, there exists a relation, analogous to (7), which expresses each v_j in terms of the derivatives which precede it if $j \leqq t$, and in terms of the t derivatives which precede it if $j>t$. In this relation, the coefficient of v_j is $(-1)^t d_1 \cdots d_t$ when $x=0$. Thus, as $v_{q+1}, \cdots, v_{q+n-1}$ vanish for $x=0$, and as they include the t derivatives which precede v_{q+n}, v_{q+n} and, then, all the derivatives which follow it, vanish for $x=0$. In other words, v is a polynomial. Thus $p_1 + \cdots + p_t$ is integral and the theorem is proved.

When the p_i are not all integers, Z_{q+1} consists of at least two terms. It is then possible to annul Z_{q+1} with b_i which are all distinct from zero, so that, by what precedes, the order of contact is at least $q+1$. In particular, when $n=2$, the order of contact is $q+1$ except when p_1 and p_2 are both integers.

COLUMBIA UNIVERSITY

[5] van der Waerden, *Einführung in die algebraische Geometrie*, Berlin, 1939, §41.

A CONVERGENCE THEOREM FOR CERTAIN LAGRANGE INTERPOLATION POLYNOMIALS

M. S. WEBSTER

In the Lagrange interpolation polynomial $L_n[f;\theta]$ where

$$
\begin{aligned}
&L_n[f;\theta] \equiv \sum_{k=1}^{n} f(x_k) l_k[\theta], \\
&l_k[\theta] \equiv l_k^{(n)}[\theta] \equiv l_k(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x - x_k)}, \\
&\phi_n(x) \equiv \prod_{k=1}^{n} (x - x_k), \\
&x = \cos\theta; \; -1 < x_k < 1; \; k = 1, 2, \cdots, n; \; n = 1, 2, \cdots,
\end{aligned} \tag{1}
$$

and $f(x)$ is a continuous function defined in $(-1, 1)$, we suppose that

$$
x_k \equiv x_k^{(n)} = \cos\theta_k = \cos k\pi/(n+1). \tag{2}
$$

Then [1],[1] we have

$$
\begin{aligned}
&\phi_n(x) = \frac{\sin(n+1)\theta}{2^n \sin\theta}, && x = \cos\theta, \\
&l_k[\theta] = \frac{(-1)^{k+1} \sin^2\theta_k \sin(n+1)\theta}{(n+1)\sin\theta(\cos\theta - \cos\theta_k)}.
\end{aligned} \tag{3}
$$

We introduce the following notations:

$$
\begin{aligned}
&t_n \equiv t \equiv \theta_1/2 \equiv \pi/2(n+1), \qquad M = \max_{-1 \leqq x \leqq 1} |f(x)|, \\
&S_k[\theta] \equiv \{l_k[\theta - t] + l_k[\theta + t]\}/2.
\end{aligned} \tag{4}
$$

We shall prove the following theorem which was suggested by a similar theorem of Grünwald [2].

THEOREM. *Let $f(x)$ be a continuous function in the interval $-1 \leqq x \leqq 1$. Then*

$$
\lim_{n \to \infty} (1/2)\{L_n[f;\theta - t_n] + L_n[f;\theta + t_n]\} = f(\cos\theta), \quad 0 < \theta < \pi, \tag{5}
$$

and the convergence is uniform in the interval $0 < \alpha \leqq \theta \leqq \pi - \alpha$ (α arbi-

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[1] The numbers in brackets refer to the bibliography.

rary, fixed constant). In general, convergence does not hold for $\theta=0$ or $\theta=\pi$.

We shall prove first that there is a constant D for which

$$\sum_{k=1}^{n} |S_k[\theta]| < D, \qquad 0 < \alpha \leqq \theta \leqq \pi - \alpha;\ n = 1, 2, \cdots. \tag{6}$$

If $\theta \neq \theta_k \pm t$, it follows from (3) and (4), by the use of trigonometric addition formulas, that

$$S_k[\theta] = \frac{(-1)^{k+1} \sin t \sin^2 \theta_k \cos (n+1)\theta}{4(n+1) \sin (\theta - t) \sin (\theta + t)} \cdot \frac{\cos \theta \cos \theta_k - \cos 2\theta \cos t}{\sin (\theta + \theta_k - t)/2 \sin (\theta - \theta_k - t)/2 \sin (\theta + \theta_k + t)/2 \sin (\theta - \theta_k + t)/2}. \tag{7}$$

If θ is restricted to the interval $0 < \alpha \leqq \theta \leqq \pi/2$, and if n is large so that $t \leqq \alpha/2$, it is easily seen (assuming $\alpha \leqq \pi/3$) that

$$\left| \frac{\sin^2 \theta_k}{\sin (1/2)(\theta + \theta_k - t) \sin (1/2)(\theta + \theta_k + t)} \right| < 4,$$

$$|S_k[\theta]| < \frac{\pi^2 C}{(n+1)^2 |\theta - \theta_k - t| \cdot |\theta - \theta_k + t|}, \tag{8}$$

$$|\csc (\theta - \lambda) \csc (\theta + \lambda)| \leqq \frac{2}{\cos \alpha - \cos 2\alpha} \equiv C,$$

$$\alpha \leqq \theta \leqq \pi/2,\ 0 \leqq \lambda \leqq \alpha/2.$$

For a given θ, there are at most two values of k for which $|\theta - \theta_k| < \pi/(n+1)$. Since $[1]\ |l_k(x)| < 2\ (-1 \leqq x \leqq 1;\ k = 1, 2, \cdots, n;\ n = 1, 2, \cdots)$, from (8) we have

$$\begin{aligned} \sum_{k=1}^{n} |S_k[\theta]| &< 4 + \sum_{1 \leqq k \leqq n,\, |\theta - \theta_k| \geqq \pi/(n+1)} |S_k[\theta]| \\ &< 4 + \frac{\pi^2 C}{(n+1)^2} \sum_{1 \leqq k \leqq n,\, |\theta - \theta_k| \geqq \pi/(n+1)} \frac{1}{|\theta - \theta_k - t| \cdot |\theta - \theta_k + t|} \\ &< 4 + \frac{2\pi^2 C}{(n+1)^2} \sum_{l=1}^{n} \left[\frac{2(n+1)}{\pi} \right]^2 \frac{1}{(2l-1)^2} \\ &< 4 + 8\pi C \sum_{m=1}^{\infty} \frac{1}{m^2} \equiv D', \qquad 0 < \alpha \leqq \theta \leqq \pi/2;\ n \geqq n_0(\alpha), \end{aligned} \tag{9}$$

because in the summation

$$|\theta - \theta_k \pm t| \geqq |\theta - \theta_k| - t \geqq l\pi/(n+1) - \pi/2(n+1), \quad l \geqq 1.$$

By continuity, (9) holds for all θ in $0<\alpha\leqq\theta\leqq\pi/2$. Since (9) is valid for n sufficiently large, there exists a D for which (6) is valid for all n if $0<\alpha\leqq\theta\leqq\pi/2$.

Since $\theta_k=\pi-\theta_{n-k+1}$, it is found that

$$l_k[\theta - t] = l_{n-k+1}[\theta' + t] \quad \text{and} \quad l_k[\theta + t] = l_{n-k+1}[\theta' - t]$$

where $\theta'=\pi-\theta$. It follows that

$$S_k[\theta] = S_{n-k+1}[\theta'], \qquad \sum_{k=1}^{n} |S_k[\theta']| < D, \qquad \pi/2 \leqq \theta' \leqq \pi - \alpha.$$

This completes the proof of (6).

From (8), if δ (>0) is fixed and if n is sufficiently large (so that $t<\delta/2$), it is seen that

$$\begin{aligned} \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|>\delta} |S_k[\theta]| &< \frac{\pi^2 C}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{(\delta - t)^2} \\ &< \frac{4\pi^2 C n}{\delta^2 (n+1)^2} = O\left(\frac{1}{n}\right). \end{aligned} \tag{10}$$

We are now ready to prove the main part of the theorem. Let θ be fixed $(0<\alpha\leqq\theta\leqq\pi-\alpha)$ and $\epsilon>0$. It is well known that

$$\sum_{k=1}^{n} l_k(x) \equiv 1, \qquad \sum_{k=1}^{n} S_k[\theta] \equiv 1.$$

Since $f(x)$ is continuous, there exists a $\delta>0$ such that

$$|f(\cos\theta) - f(\cos\theta_k)| < \epsilon \quad \text{provided} \quad |\theta - \theta_k| \leqq \delta.$$

Let

$$\begin{aligned} \Delta &= (1/2)\{L_n[f;\theta - t] + L_n[f;\theta + t]\} - f(\cos\theta) \\ &= \sum_{k=1}^{n} \{f(\cos\theta_k) - f(\cos\theta)\} S_k[\theta] \\ &= \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|\leqq\delta} \{f(\cos\theta_k) - f(\cos\theta)\} S_k[\theta] \\ &\quad + \sum_{1\leqq k\leqq n,\,|\theta-\theta_k|>\delta} \{f(\cos\theta_k) - f(\cos\theta)\} S_k[\theta]. \end{aligned}$$

Then, by the use of (6) and (10), for sufficiently large n, we find that

$$|\Delta| < \epsilon \sum_{1\leq k\leq n,\,|\theta-\theta_k|\leq\delta} |S_k[\theta]| + 2M \sum_{1\leq k\leq n,\,|\theta-\theta_k|>\delta} |S_k[\theta]|$$

$$< \epsilon D + 2MO\left(\frac{1}{n}\right) < (D+1)\epsilon, \qquad n > N.$$

As in Rogosinki's theorem for Fourier series, the theorem may be easily extended so that convergence holds at any point x ($\neq \pm 1$) of continuity of a bounded function $f(x)$, convergence being uniform in any closed interval of continuity (excluding $x = \pm 1$). In addition, t_n may be replaced by $p\pi/2(n+1)$ where p is any fixed odd integer.

Since $L_n[f; -t] = L_n[f; t]$, the theorem would involve (for $\theta = 0$) the convergence of $L_n[f; t]$ to $f(\cos 0)$. This convergence does not hold for all continuous $f(x)$ because

$$\lambda_n \equiv \sum_{k=1}^{n} |l_k(1)| = n,$$

and, according to H. Hahn [3], a necessary and sufficient condition for $L_n[f; 0]$ to converge to $f(\cos 0)$ for all continuous $f(x)$ is that λ_n be bounded for all n. We give an example (compare [4] and [5]) of a continuous function for which convergence does not occur at $x = 1$.

Let $f_n(x)$ be defined for each n ($n = 1, 2, \cdots$) as follows:

$$(11)\quad f_n(x) = \begin{cases} (-1)^{n-1}, & -1 \leqq x < x_n, \\ (-1)^k\left[1 - \dfrac{2(x-x_{k+1})}{x_k - x_{k+1}}\right], & x_{k+1} \leqq x \leqq x_k;\ k = 1, 2, \cdots, n-1, \\ 1, & x_1 < x \leqq 1. \end{cases}$$

Now,

$$L_n[f_n; t] = -\sum_{k=1}^{n} (-1)^k l_k[t]$$

$$= \frac{2}{(n+1)\sin t} \sum_{k=1}^{n} \frac{\sin^2 kt \cos^2 kt}{\sin (k+1/2)t \sin (k-1/2)t}.$$

Since

$$\frac{\sin kt}{\sin (k+1/2)t} > \frac{2}{3}, \qquad k = 1, 2, \cdots, n,$$

we have

$$(12)\quad |L_n[f_n; t]| > \frac{4}{3(n+1)\sin t} \sum_{k=1}^{n} \cos^2 kt$$

$$> \frac{2}{3(n+1)\sin t} \sum_{k=1}^{n} (1 + \cos\theta_k) > \frac{4n}{3\pi}.$$

By the Weierstrass approximation theorem, we may approximate $f_n(x)$ by a polynomial $g_n(x)$ of degree $d(n)$ such that in $(-1, 1)$,

$$|g_n(x)| \leqq 3/2, \qquad |L_n[g_n; t]| > 2n/3\pi. \tag{13}$$

This is possible, in view of (12), because

$$|L_n[f_n; t] - L_n[g_n; t]| = \left| \sum_{k=1}^{n} [f_n(x_k) - g_n(x_k)] l_k[t] \right| \leqq \epsilon'$$

provided

$$|f_n(x) - g_n(x)| \leqq (\epsilon' \Big/ \sum_{k=1}^{n} |l_k[t]|), \qquad -1 \leqq x \leqq 1.$$

Let

$$g(x) \equiv \sum_{i=1}^{\infty} c_i g_{n_i}(x), \qquad -1 \leqq x \leqq 1, \tag{14}$$

where $c_1 = n_1 = 1$, and

$$c_{i+1} = \min\left\{\frac{c_i}{4}, \frac{1}{T_i}\right\}, \quad T_i \equiv \sum_{k=1}^{n_i} |l_k^{(n_i)}[t_{n_i}]|, \qquad i = 1, 2, \cdots; \tag{15}$$

n_i is the smallest integer satisfying the conditions

$$\begin{aligned} &\text{(a)} \quad n_i \geqq d(n_{i-1}) + 1, \\ &\text{(b)} \quad |c_i L_{n_i}[g_{n_i}; t_{n_i}] - 8| > 4^i, \qquad i = 2, 3, \cdots. \end{aligned} \tag{16}$$

Condition (b) is possible because of (13). From (15), it follows that

$$c_{i+1} \leqq 1/4^i, \qquad i = 0, 1, 2, \cdots, \tag{17}$$

and the series for $g(x)$ converges uniformly so that $g(x)$ is continuous and $|g(x)| \leqq 2$ in $(-1, 1)$. Let $m = n_r$ where r $(\neq 1)$ is a positive integer. Using (15) and (17), we have

$$\begin{aligned} &\left| \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] \right| \leqq \sum_{i=r+1}^{\infty} c_i \frac{3}{2} \sum_{k=1}^{m} |l_k^{(m)}[t_m]| \leqq 2, \\ &\left| \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| \leqq 2. \end{aligned} \tag{18}$$

If $w(x)$ is any polynomial of degree less than m in $x = \cos\theta$, then $L_m[w; \theta] \equiv w(x)$. Since

$$\sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m]$$

$$= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i L_m[g_{n_i}; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m]$$

$$= c_r L_m[g_m; t_m] + \sum_{i=1}^{r-1} c_i g_{n_i}(\cos t_m) + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m],$$

we find from (16) and (18) that

$$\begin{aligned} &| L_m[g; t_m] - g(1) | \\ &\quad = | \{L_m[g; t_m] - g(\cos t_m)\} + \{g(\cos t_m) - g(1)\} | \\ &\quad \geqq \left| \sum_{i=1}^{\infty} c_i L_m[g_{n_i}; t_m] - g(\cos t_m) \right| - 4 \\ &\quad \geqq \left| c_r L_m[g_m; t_m] + \sum_{i=r+1}^{\infty} c_i L_m[g_{n_i}; t_m] - \sum_{i=r}^{\infty} c_i g_{n_i}(\cos t_m) \right| - 4 \\ &\quad > 4^r, \qquad r = 2, 3, \cdots . \end{aligned}$$

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PURDUE UNIVERSITY

A NOTE ON COMPLEMENTARY SUBSPACES IN A RIEMANNIAN SPACE

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1. Introduction.[2] Let

$$(1.1)\qquad x^\kappa = x^\kappa(u^a), \quad \kappa, \lambda, \mu, \nu = 1, \cdots, n;\ a, b, \cdots, f = 1, \cdots, m,$$

be the equations of a V_m in a V_n with fundamental tensor $g_{\lambda\kappa}$ and let

$$(1.2)\qquad B_a^\kappa = \partial_a x^\kappa \equiv \frac{\partial x^\kappa}{\partial u^a}.$$

Then the fundamental tensor and curvature tensor of V_m in V_n are, respectively,

$$(1.3)\qquad {}'g_{cb} = g_{\lambda\kappa} B_c^\lambda B_b^\kappa,$$

$$(1.4)\qquad H_{cb}^{\cdot\cdot\kappa} = D_c B_b^\kappa \equiv \partial_c B_b^\kappa + \Gamma_{\mu\lambda}^\kappa B_c^\mu B_b^\lambda - {}'\Gamma_{cb}^a B_a^\kappa,$$

where D denotes the generalized covariant differentiation with respect to V_m in V_n; and $\Gamma_{\mu\lambda}^\kappa$ and ${}'\Gamma_{cb}^a$ are, respectively, the Christoffel symbols of the second kind for V_n and V_m.

By definition a V_m in V_n is said to be *totally semi-umbilical*[3] in V_n if a vector v_κ exists such that

$$(1.5)\qquad v_\kappa H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb}$$

is satisfied at every point of V_m. In particular, this condition is evidently fulfilled when $H_{cb}^{\cdot\cdot\kappa}$ has the form $H_{cb}^{\cdot\cdot\kappa} = {}'g_{cb} n^\kappa$, n^κ being a certain vector; in this case we call V_m *totally umbilical* in V_n.

In what follows we shall consider the subspaces V_m: $x^p = \text{const.}$ in a V_n with fundamental tensor of the form

$$(1.6)\qquad g_{\lambda\kappa} = \begin{pmatrix} g_{cb} & 0 \\ 0 & g_{qp} \end{pmatrix}, \qquad \begin{matrix} a, b, \cdots, f = 1, \cdots, m, \\ p, q, \cdots, s = m+1, \cdots, n. \end{matrix}$$

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[1] The author is a Chinese Ying-Keng Funds student. He wishes to thank Professor D. J. Struik for the conversations they had from time to time during his stay at Cambridge, Massachusetts.

[2] For the theory of subspaces V_m in a Riemannian n-space V_n, see Schouten-Struik, *Einführung in der neuern Methoden der Differentialgeometrie* II, Groningen, 1938 chap. 3.

[3] D. Perepelkine, *Sur la courbure et les espaces normaux d'une V_m dans R_n*, Rec. Math. (Mat. Sbornik) N.S. vol. 42 (1935) pp. 81–100.

The two families of subspaces V_m: $x^p=\text{const.}$ and V_{n-m}: $x^a=\text{const.}$ are called *completementary families* of subspaces in V_n. Recently, Yano[4] proved that a condition for V_m to be totally umbilical in V_n is that g_{cb} be of the form[5] $g_{cb}=\sigma(x^\epsilon)\bar{g}_{cb}(x^a)$. We shall obtain, among other results, a similar condition for V_m to be totally semi-umbilical in V_n.

2. **First normal complex.** Let w^a and y^a be two arbitrary vectors in V_m. Then the vector $y^c w^b H_{cb}^{\cdot\cdot\kappa}$ spans the first normal complex of V_m in V_n, whose dimensionality m_1 is therefore equal to the rank of the matrix $[H_{cb}^{\cdot\cdot\kappa}]$. In this matrix, as well as in every matrix appearing hereafter, κ or p indicates the column and the combination of $b, c, \cdots$ the row.

Now for the subspaces V_m: $x^p=\text{const.}$ in a V_n with fundamental tensor (1.6), we have

$$B_a^\kappa = \frac{\partial x^\kappa}{\partial x^a} = \delta_a^\kappa, \qquad {}'g_{cb} = g_{cb}, \tag{2.1}$$

$$H_{cb}^{\cdot\cdot a} = \Gamma_{cb}^a - {}'\Gamma_{cb}^a, \qquad H_{cb}^{\cdot\cdot p} = \Gamma_{cb}^p. \tag{2.2}$$

But from (1.6) and the definition of the Christoffel symbols of the second kind

$$\Gamma_{\mu\lambda}^\kappa = (1/2)g^{\kappa\nu}(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\nu g_{\mu\lambda})$$

it follows at once that

$$\begin{aligned} &\Gamma_{cb}^a = {}'\Gamma_{cb}^a, && \Gamma_{cb}^p = -(1/2)g^{pq}\partial_q g_{cb}, \\ &\Gamma_{cq}^a = (1/2)g^{ab}\partial_q g_{cb}, && \Gamma_{cr}^p = (1/2)g^{pq}\partial_c g_{rq}. \end{aligned} \tag{2.3}$$

Thus (2.2) become

$$H_{cb}^{\cdot\cdot a} = 0, \qquad H_{cb}^{\cdot\cdot p} = -(1/2)g^{pq}\partial_q g_{cb}. \tag{2.4}$$

And therefore the dimensionality m_1 of the first normal complex of V_m in V_n is equal to the rank of the matrix $[g^{pq}\partial_q g_{cb}]$. Since $\text{Det}\,(g^{pq})\neq 0$, m_1 is also the rank of the matrix $[\partial_p g_{cb}]$. Hence[6] there

[4] K. Yano, *Conformally separable quadratic differential forms*, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 83–86. For $n=m+1$ see L. P. Eisenhart, *Riemannian geometry*, Princeton, 1926 p. 182.

[5] Throughout this paper we denote by $\rho, \sigma, \theta, \phi$ scalar functions of x^ϵ.

[6] See, for example, T. Levi-Civita, *The absolute differential calculus*, London, 1927 pp. 9–12.

exist m_1 components of g_{cb}, which are functionally independent with regard to x^p, such that each component of g_{cb} is expressible in terms of them and x^a. Conversely, it is evident from (2.4) that if g_{cb} has this property, the first normal complex of V_m in V_n is of dimension m_1. Hence we have this theorem.[7]

THEOREM 2.1. *The first normal complex of the subspaces $x^p=$const. in a V_n with fundamental tensor* (1.6) *is of dimension m_1, if and only if the matrix $[\partial_p g_{cb}]$ is of rank m_1, that is if g_{cb} is of the form $g_{cb}=g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1})$, where the ρ's are m_1 functions of x^s which are functionally independent with regard to x^p.*

Now it follows from (2.4) and Theorem 2.1 that the components of the vector $y^c w^b H_{cb}^{\cdots\kappa}$, which spans the first normal complex, are

$$y^c w^b H_{cb}^{\cdots a} = 0,$$

$$(2.5) \qquad y^c w^b H_{cb}^{\cdots p} = -(1/2) y^c w^b g^{pq} \left(\frac{\partial g_{cb}}{\partial \rho_1} \partial_q \rho_1 + \cdots + \frac{\partial g_{cb}}{\partial \rho_{m_1}} \partial_q \rho_{m_1} \right).$$

To see the implication of these equations let us consider a certain fixed V_{n-m}: $x^a=x_0^a$. Each V_m of the family $x^p=$const. has a point in common with V_{n-m}, at which the first normal complex of V_m lies in the tangent space of V_{n-m}. Equations (2.5) then show that these first normal complexes are orthogonal to the subspaces

$$\rho_1(x_0^a, x^p) = \text{const.}, \cdots, \rho_{m_1}(x_0^a, x^p) = \text{const.}$$

of V_{n-m}.

THEOREM 2.2. *If a V_n admits two complementary families of V_m and V_{n-m}, then the first normal complexes, dimensionality m_1, of V_m at points of any fixed V_{n-m} are orthogonal to a family of subspaces V_{n-m-m_1} in V_{n-m}.*

The condition for V_m to be minimal in V_n is $'g^{cb}H_{cb}^{\cdots\kappa}=0$, which, by (2.1) and (2.4), can be written $g^{cb}\partial_p g_{cb}=0$, that is, $\partial_p \operatorname{Det}(g_{cb})=0$. Hence this theorem follows.[8]

THEOREM 2.3. *If a V_n admits two complementary families of V_m and V_{n-m}, a necessary and sufficient condition for V_m to be minimal in V_n is that V_{n-m} determine a correspondence between them which preserves volume.*

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[7] For $m_1=0$, see Eisenhart, loc. cit. p. 186 Example 13.

[8] For $n=m+1$, see Eisenhart, loc. cit. p. 179.

3. **Totally semi-umbilical** V_m. According to (1.5), (2.1) and (2.4), the condition for $x^p=\text{const.}$ to be totally semi-umbilical in V_n is that a vector v^p in V_{n-m} exists such that $v^p\partial_p g_{cb}=g_{cb}$, that is,

$$v^p\partial_p \log g_{cb} = 1. \tag{3.1}$$

From this it follows that

$$v^p\partial_p \log (g_{cb}/g_{ed}) = 0. \tag{3.2}$$

Now if the first normal complex of V_m in V_n is of dimension m_1, then by Theorem 2.1 g_{cb} is of the form

$$g_{cb} = g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}). \tag{3.3}$$

Consequently, (3.2) gives

$$\theta_1 \frac{\partial}{\partial\rho_1} \log (g_{cb}/g_{ed}) + \cdots + \theta_{m_1} \frac{\partial}{\partial\rho_{m_1}} \log (g_{cb}/g_{ed}) = 0, \tag{3.4}$$

where

$$\theta_1 = v^p\partial_p\rho_1, \cdots, \theta_{m_1} = v^p\partial_p\rho_{m_1}. \tag{3.5}$$

Conversely, let m_1 functions $\theta_1, \cdots, \theta_{m_1}$ exist satisfying (3.4). Then since $\rho_1, \cdots, \rho_{m_1}$ are independent with regard to x^p, the matrix $[\partial_p\rho_1, \cdots, \partial_p\rho_{m_1}]$ is of rank m_1. Therefore, the system of linear equations (3.5) has solutions for v^p; that is, v^p exist satisfying (3.2) and also (3.1). Hence, when (3.3) is true, (3.4) is a necessary and sufficient condition for V_m to be totally semi-umbilical in V_n.

On the other hand, by a well known theorem[9] on the essential parameters of a set of functions, equation (3.4) is also the condition that there exist m_1-1 functions $\sigma_1, \cdots, \sigma_{m_1-1}$ of x^a and the ρ's (and therefore of x^s) such that g_{cb}/g_{ed} is expressible in terms of them and x^a; that is, that g_{cb} is of the form

$$g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}). \tag{3.6}$$

It is seen that $\bar{g}_{cb}$ cannot be expressed in terms of x^a and less than m_1-1 independent (with regard to x^p) functions σ's; otherwise, g_{cb} would be expressible in terms of x^a and less than m_1 functions, and consequently by Theorem 2.1, the first normal complex of V_m in V_n would be of dimension less than m_1.

THEOREM 3.1. *In a V_n with fundamental tensor* (1.6), *each of the subspaces $x^p=\text{const.}$, whose first normal complexes are of dimension*

[9] See, for example, L. P. Eisenhart, *Continuous groups of transformation*, Princeton, 1933, p. 9.

m_1, is totally semi-umbilical in V_n, if and only if g_{cb} is of the form

$$g_{cb} = \sigma_{m_1}\bar{g}_{cb}(x^a, \sigma_1, \cdots, \sigma_{m_1-1}),$$

where the σ's are m_1 functions of x^κ which are independent with regard to x^p.

For $m_1=1$, we have Yano's result quoted in §1.

4. **Normal complexes of higher order.** We now return to the end of §2 and consider the matrix

$$\begin{pmatrix} B_a^{\kappa} \\ H_{cb}^{\cdots\kappa} \\ D_f H_{ed}^{\cdots\kappa} \end{pmatrix}. \tag{4.1}$$

Let m_1 and m_2 be, respectively, the dimensionalities of the first and second normal complexes of V_m in V_n, then the rank of the above matrix is $m+m_1+m_2$. Taking account of (2.3), (2.4) and

$$D_f H_{ed}^{\cdots\kappa} = \partial_f H_{ed}^{\cdots\kappa} + \Gamma_{\mu\lambda}^{\kappa} H_{ed}^{\cdots\lambda} B_f^{\mu} - {}'\Gamma_{fd}^{c} H_{ec}^{\cdots\kappa} - {}'\Gamma_{fe}^{c} H_{cd}^{\cdots\kappa}$$

we can easily prove from (4.1) that the following matrices are all of rank m_1+m_2:

$$\begin{pmatrix} H_{cb}^{\cdots p} \\ D_f H_{ed}^{\cdots p} \end{pmatrix}, \quad \begin{pmatrix} H_{cb}^{\cdots p} \\ \partial_f H_{ed}^{\cdots p} + \Gamma_{fq}^{p} H_{ed}^{\cdots q} \end{pmatrix}, \quad \begin{pmatrix} g^{pq}\partial_q g_{cb} \\ g^{pq}\partial_f\partial_q g_{ed} + (1/2)(\partial_f g^{pq})\partial_q g_{ed} \end{pmatrix},$$

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_f\partial_p g_{ed} + (1/2)g_{pq}(\partial_f g^{qr})\partial_r g_{ed} \end{pmatrix}. \tag{4.2}$$

The last matrix shows that, unlike m_1, the dimensionality m_2 of the second normal complex of V_m in V_n depends not only on the nature of g_{cb} but also on that of g_{qp}.

If $g_{qp}=g_{qp}(x^r)$, that is, if the complementary V_{n-m} are totally geodesic in V_n (cf. Theorem 2.1 for $m_1=0$), the matrix (4.2) reduces to

$$\begin{pmatrix} \partial_p g_{cb} \\ \partial_p\partial_f g_{ed} \end{pmatrix}.$$

This matrix is of rank m_1+m_2, and therefore g_{cb}, $\partial_f g_{ed}$ can be expressed in terms of x^a and m_1+m_2 (but not less) functions of x^κ which are independent with regard to x^p. But

$$\begin{aligned} g_{cb} &= g_{cb}(x^a, \rho_1, \cdots, \rho_{m_1}), \\ \partial_f g_{ed} &= \phi_0 + \phi_1\partial_f\rho_1 + \cdots + \phi_{m_1}\partial_f\rho_{m_1}, \end{aligned} \tag{4.3}$$

where the ϕ's are some functions of x^a and the ρ's. Therefore the first and second normal complexes of V_m in V_n are of dimension m_1 and m_2, if and only if (4.3) is true and $\partial_f\rho_1, \cdots, \partial_f\rho_{m_1}$ are expressible in terms of x^a, $\rho_1, \cdots, \rho_{m_1}$, and m_2 other functions $\rho_{m_1+1}, \cdots, \rho_{m_1+m_2}$, which, together with $\rho_1, \cdots, \rho_{m_1}$, form m_1+m_2 functions independent with regard to x^p.

This being the case, we have

$$\partial_p g_{cb} = \frac{\partial g_{cb}}{\partial \rho_1}\partial_p\rho_1 + \cdots + \frac{\partial g_{cb}}{\partial \rho_{m_1}}\partial_p\rho_{m_1},$$

$$\partial_p\partial_f g_{ed} = \frac{\partial\partial_f g_{ed}}{\partial \rho_1}\partial_p\rho_1 + \cdots + \frac{\partial\partial_f g_{ed}}{\partial \rho_{m_1+m_2}}\partial_p\rho_{m_1+m_2}.$$

But if w^a, y^a, z^a are three arbitrary vectors in V_m, the vectors $y^c w^b H_{cb}^{\cdot\cdot\rho}$ and $z^f y^e w^d D_f H_{ed}^{\cdot\cdot\rho}$ span the first two normal complexes of V_m in V_n. Therefore by an argument similar to that which led to Theorem 2.2, we conclude that the first two normal complexes of V_m at points of a fixed V_{n-m} are orthogonal to a family of $V_{n-m-m_1-m_2}$ in V_{n-m}.

The above result can easily be extended to cover the normal complexes of higher order of V_m in V_n; indeed we have the following two theorems.

THEOREM 4.1. *In a V_n with fundamental tensor*

$$g_{\lambda\varkappa} = \begin{pmatrix} g_{cb}(x^z) & 0 \\ 0 & g_{qp}(x^r) \end{pmatrix},$$

the normal complexes of the subspaces $x^p = const.$ are of dimension $m_1, m_2, \cdots$ if and only if the matrices

$$\begin{pmatrix} \partial_p g_{cb} \end{pmatrix}, \begin{pmatrix} \partial_p g_{cb} \\ \partial_p\partial_f g_{ed} \end{pmatrix}, \cdots$$

are of ranks m_1, $m_1+m_2, \cdots$, respectively.

THEOREM 4.2. *If a V_n admits two families of complementary V_m and V_{n-m} and if V_{n-m} are totally geodesic in V_n, then the first l ($l=1, 2, \cdots$) normal complexes of V_m at points of any fixed V_{n-m} are orthogonal to a family of $V_{n-m-m_1-m_2\cdots-m_l}$ in V_{n-m}.*

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ON THE JOIN OF TWO COMPLEXES

C. E. CLARK

1. **Introduction.** In this note we point out an isomorphism between the $(r+1)$-dimensional Betti group of the join (defined below) of two complexes and a subgroup of the r-dimensional Betti group of the product of the two complexes. Using this isomorphism the Betti groups of the join are derived from those of the product in case the complexes are finite.[1]

2. **Definition of the join (K_1, K_2) of K_1 and K_2.** To define the join of two complexes we first define the join (σ, τ) of a p-dimensional simplex σ and a q-dimensional simplex τ, $p, q = 0, 1, \cdots$. This join is a $(p+q+1)$-dimensional simplex with a p-dimensional side associated with σ and the opposite side, which is q-dimensional, associated with τ. These sides will not be distinguished from σ and τ, respectively. Now consider the complexes K_1 and K_2. Consider the set consisting of the simplexes σ_α of K_1, the simplexes τ_β of K_2, and the simplexes $(\sigma_\alpha, \tau_\beta)$. In a natural way this set forms a complex. We define the *join* (K_1, K_2) of K_1 and K_2 to be the first barycentric subdivision of this complex.

3. **The rays.** By the rays of (σ, τ) we mean the straight line segments each of which joins a point of σ and a point of τ. These rays cover (σ, τ). Also no two rays intersect except possibly at an end point. The rays of all $(\sigma_\alpha, \tau_\beta)$ of (K_1, K_2) are called the *rays*.

Let N_i, $i=1, 2$, be the subcomplex made up of the simplexes of (K_1, K_2) that have at least one vertex in K_i together with the faces of all such simplexes. It is known that each ray meets the intersection $N_1 \cap N_2$ in exactly one point.[2] Furthermore N_i and $N_1 \cap N_2$ can be homotopically deformed in N_i along the rays into K_i, $i=1, 2$.[2] It follows that $N_1 \cap N_2$ and the product $K_1 \times K_2$ are homeomorphic (the complexes being considered as point sets).

4. **The theorem.** We prove this theorem.

THEOREM 1. *There is an isomorphism between the $(r+1)$-dimensional*

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[1] The Betti groups of the join of two finite complexes are known. They were computed by H. Freudenthal in his paper *Die Bettischen Gruppen der Verbindung Zweier Polytope*, Fund. Math. vol. 29 (1937) pp. 145–150.

[2] For a proof see our paper *Simultaneous invariants of a complex and subcomplex*, Duke Math. J. vol. 5 (1939) pp. 62–71.

Betti group of (K_1, K_2) *and the subgroup of those homology classes of the* r*-dimensional Betti group of* $N_1 \cap N_2$ *which contain cycles that bound both in* N_1 *and* N_2, $r \geqq 0$; *all chains considered in this paper are finite and have integral coefficients.*

In this theorem K_1 and K_2 may be infinite.

PROOF. We note that $(K_1, K_2) = N_1 + N_2$. Hence we know that there is a homomorphism from the $(r+1)$-dimensional Betti group of $N_1 + N_2$ onto those homology classes of the r-dimensional Betti group of $N_1 \cap N_2$ which contain cycles that bound both in N_1 and N_2, $r \geqq 0$; furthermore, the kernel of this homomorphism consists of the homology classes that contain Summenzyklen, that is, cycles that are equal to the sum of two cycles, one in N_1 and the other[3] in N_2. To prove Theorem 1 we shall show that a Summenzyklus bounds in (K_1, K_2). To prove this, consider a cycle Z of N_1. This cycle can be homotopically deformed in N_1 along the rays into a singular cycle of K_1. We know that (K_1, K_2) contains the join of K_1 and an arbitrary point of K_2. Hence any cycle of K_1 can be homotopically deformed in (K_1, K_2) into a cycle of a vertex of K_2. Since the dimension of Z is greater than zero, it follows that Z is homologous to zero in (K_1, K_2). Similarly, a cycle of N_2 with dimension greater than zero bounds in (K_1, K_2).

5. **Some properties of** $N_1 \cap N_2$. From now on K_1 and K_2 are finite. For any chain A of a complex let $|A|$ denote the closure of the set of those simplexes at which[4] $A \neq 0$. We know that with each pair of chains $C_1 \subset K_1$ and $C_2 \subset K_2$ there is associated a chain $C_1 \times C_2 \subset N_1 \cap N_2$; the dimension of $C_1 \times C_2$ is the sum of the dimensions of C_1 and C_2; and $|C_1 \times C_2| = |C_1| \times |C_2|$.[5] It follows that the projection of $|C_1 \times C_2|$ into K_1 or K_2 along the rays is a subset of $|C_1|$ or $|C_2|$, respectively. Also if C_2 is a zero-dimensional cycle, and $C_2 = 0$ at all but one vertex of K_2 at which $C_2 = 1$, then $C_1 \times C_2$ and C_1 are isomorphic, and homotopic deformation of $C_1 \times C_2$ along the rays shows that $C_1 \times C_2 \sim C_1$ in N_1 (all the rays used in the deformation meet at a point of K_2).

Let

$$Z_i^p \subset K_1 \tag{1}$$

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[3] See Alexandroff-Hopf, *Topologie*. I, p. 293, Theorem V. This theorem and its applicability were pointed out by the referee.

[4] An r-dimensional chain is a function defined over all r-dimensional simplexes of a complex.

[5] These and the following properties of $N_1 \cap N_2 = K_1 \times K_2$ are proved in Alexandroff-Hopf, *Topologie*. I, pp. 299–310.

and

$$Z_j^q \subset K_2 \tag{2}$$

be homology bases[6] for the p-dimensional cycles of K_1 and the q-dimensional cycles of K_2. We know that there is a set of s-dimensional chains $u_k^s \subset K_1$, $s=2, 3, \cdots$, in $(1=1)$ correspondence with the $(s-1)$-dimensional torsion coefficients of K_1, and there is a set of t-dimensional chains $v_l^t \subset K_2$, $t=2, 3, \cdots$, in $(1=1)$ correspondence with the $(t-1)$-dimensional torsion coefficients of K_2 such that a homology basis for the r-dimensional cycles of $N_1 \cap N_2$ is given by

$$Z_{ij}^r = Z_i^p \times Z_j^q, \quad p+q=r, \text{ order of } Z_{ij}^r \neq 1, \tag{3}$$

and

$$C_{kl}^r = c_{kl}^r (u_k^s \times v_l^t)^*, \quad s+t=r+1, c_{kl}^r \neq 1, \tag{4}$$

where c_{kl}^r is the reciprocal of the greatest common divisor of the torsion coefficients associated with u_k^s and v_l^t. Furthermore the order of Z_{ij}^r is the greatest common divisor of the orders[7] of Z_i^p and Z_j^q, and the order of C_{kl}^r is the reciprocal of c_{kl}^r.

6. **The cycles of $N_1 \cap N_2$ that bound both in N_1 and N_2.** We choose a cycle from (1) with $p=0$ and a cycle from (2) with $q=0$. Each of these cycles will be denoted by the same symbol Z_1^0.

THEOREM 2. *A homology basis for the subgroup mentioned in Theorem* 1 *consists of the following subset of* (3) *and* (4): *if $r>0$, the basis consists of Z_{ij}^r with both $p>0$ and $q>0$, $Y_{ij}^r = Z_i^r \times (Z_1^0 - Z_j^0)$, $j \neq 1$, $X_{ij}^r = (Z_1^0 - Z_i^0) \times Z_j^r$, $i \neq 1$, and all C_{ij}^r; if $r=0$, the basis consists of $W_{ij}^0 = (Z_1^0 - Z_i^0) \times (Z_1^0 - Z_j^0)$, $i \neq 1$, $j \neq 1$.*

PROOF. In the first place consider Z_{ij}^r with both $p \neq 0$ and $q \neq 0$. We know that Z_{ij}^r can be homotopically deformed in N_1 along the rays into $|Z_i^p|$, a p-dimensional subcomplex of K_1. Since $p<r$, this implies that Z_{ij}^r bounds in N_1. From symmetry, if $p \neq 0$ and $q \neq 0$, the cycle Z_{ij}^r bounds both in N_1 and N_2.

Consider next C_{kl}^r. Since $s+t=r+1$, $s>1$, and $t>1$, it follows that

[6] By a homology basis for the r-dimensional cycles of a complex we mean a set of cycles obtained by expressing the r-dimensional Betti group as the usual direct sum of free cyclic groups and finite cyclic groups whose orders are the torsion coefficients and by choosing a cycle from a generator of each summand.

[7] By the order of a cycle we mean the order of its homology class. Also it is understood that a free group has order zero and that the greatest common divisor of zero and a positive integer is that integer.

$s<r$ and $t<r$. Hence C_{kl}^r bounds both in N_1 and N_2 because C_{kl}^r is homotopic in N_1 to a cycle of $|u_k^s|$ and is homotopic in N_2 to a cycle of $|v_l^t|$.

Next consider Z_{ij}^r with $p=r\neq 0$. We can assume that $Z_j^0=0$ at all but one vertex of K_2 and that $Z_j^0=1$ at the exceptional vertex. Then as shown above $Z_{ij}^r\sim Z_i^r$ in N_1. From this we shall deduce that Z_{ij}^r does not bound in N_1. Suppose Z_{ij}^r does bound in N_1. Then $Z_i^r=\dot{F}$, $F\subset N_1$. We deform F in N_1 along the rays into the singular chain $F'\subset K_1$. The singular cycle $Z_i^r=\dot{F}'$ which contradicts the definition of Z_i^r.

From symmetry we see that Z_{ij}^r with $q=r\neq 0$ does not bound in N_2.

Now consider Y_{ij}^r. Since $Z_i^r\times Z_1^0$ and $Z_i^r\times Z_j^0$ are each homologous in N_1 to Z_i^r, it follows that Y_{ij}^r bounds in N_1. That Y_{ij}^r also bounds in N_2 follows from the fact that Y_{ij}^r can be homotopically deformed into a complex consisting of two vertices. From symmetry we know that X_{ij}^r bounds both in N_1 and N_2.

Finally if $r=0$, we see that W_{ij}^0 bounds both in N_1 and N_2.

The theorem follows easily from these facts.

7. **The Betti groups of** (K_1, K_2). Theorems 1 and 2 imply this theorem.

FREUDENTHAL'S THEOREM.[1] *From* (1) *delete one cycle with* $p=0$ *and from* (2) *delete one cycle with* $q=0$; *each association of one of the remaining* Z_i^p *with one of the remaining* Z_j^q, $p+q=r$, *represents a generator of the* $(r+1)$*-dimensional Betti group of* (K_1, K_2); *the order of this generator is the greatest common divisor of the orders of* Z_i^p *and* Z_j^q; *furthermore each* C_{kl}^r *in* (4) *represents a generator of order* $1/c_{kl}$; *all such generators with their orders define the* $(r+1)$*-dimensional Betti group of* (K_1, K_2).

8. **The case** $r=-1$. The join (K_1, K_2) is connected because two points of K_i can be joined to an arbitrary point of K_j, $j\neq i$.

PURDUE UNIVERSITY

A MIXED BOUNDARY VALUE PROBLEM

SOME REMARKS ON A PROBLEM OF A. WEINSTEIN

ALBERT E. HEINS

At present the unilateral Laplace transform has had many interesting applications. To cite several types we have (a) initial value problems in ordinary differential equations, (b) initial value and boundary value problems in partial differential equations with one space variable, (c) "initial value problems" in ordinary difference equations, and (d) "initial value" and "boundary value problems" in partial difference equations.

Titchmarsh[1] and his collaborators, Cooper[2] and Busbridge,[3] have indicated that much can be done with the finite Laplace transform, that is,

$$\int_a^b e^{-sx} f(x)dx = g(s). \tag{1}$$

The transform (1) contains as a special case a finite Fourier transform, first used by Stokes in 1850 for the solution of certain boundary value problems in mathematical physics. The finite Fourier transform has been recently revived by Doetsch[4] and Kniess.[5] It cannot be applied as widely as (1), since it assumes, a priori, that the boundary value problem naturally has a Fourier series as an expansion. On the other hand (1) slips quite neatly into an expansion which is natural to the boundary value problem.

By applying a transformation of the type (1) to a linear differential equation (ordinary or partial) with constant coefficients under given boundary conditions, boundary functions which are superfluous are introduced. Picone and others have demonstrated that by solving the reduced equation and noting the fact that $g(s)$ is an entire function of the parameter s, these superfluous boundary elements may be eliminated. This procedure may become exceedingly difficult to carry through. The method we employ here makes use of a regularity condition which is introduced by rendering the boundary conditions symmetric.

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[1] Titchmarsh, J. London Math. Soc. vol. 14 (1939) p. 118.

[2] Cooper, J. London Math. Soc. vol. 14 (1939) p. 124.

[3] Busbridge, J. London Math. Soc. vol. 14 (1939) p. 128.

[4] Doetsch, Math. Ann. vol. 112 (1935) p. 52.

[5] Kniess, Math. Zeit. vol. 44 (1939) p. 266.

We propose to treat here a mixed boundary value problem in potential theory with the aid of (1). This problem was first investigated by A. Weinstein in 1927 and later by Cooper,[6] Bochner and Poritsky. The author wishes to thank Dr. Alexandre Weinstein for having brought this problem to his attention and for having discussed it with him.

We are concerned here with the following problem. Find the conditions under which the solution of the harmonic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2}$$

under the mixed boundary conditions

$$u(x, 0) = 0, \qquad u(x, 1) = ku_y(x, 1)$$

may be bounded or of finite exponential order for $-\infty < x < \infty$. We modify this problem by considering the solution of (2) which is odd and continuous in y and which satisfies the mixed boundary conditions

$$u(x, 1) = ku_y(x, 1), \qquad -u(x, -1) = ku_y(x, -1).$$

We write

$$g(x, s) = \int_{-1}^{1} e^{-sy} u(x, y)\, dy.$$

Since $u(x, y)$ is a harmonic function we may differentiate twice under the integral sign with respect to x, and we have

$$\frac{\partial^2 g}{\partial x^2} = -\int_{-1}^{1} e^{-sy} \frac{\partial^2 u}{\partial y^2}\, dy.$$

Upon integrating by parts and noting the boundary conditions of the modified problem we have

$$\begin{aligned} \frac{\partial^2 g}{\partial x^2} + s^2 g &= -e^{-s} u_y(x, 1) + e^{s} u_y(x, -1) - s\left[e^{-s} u(x, 1) - e^{-s} u(x, -1)\right] \\ &= u_y(x, 1)\left[(e^{s} - e^{-s}) - sk(e^{s} + e^{-s})\right]. \end{aligned}$$

But

$$u_y(x, 1) = \frac{(\partial^2 g/\partial x^2) + s^2 g}{e^{s} - e^{-s} - sk(e^{s} + e^{-s})} \tag{3}$$

[6] Loc. cit.

and (3) is finite for all s. Hence

$$g_{zz}(s_j) + s_j^2 g(s_j) = 0$$

for all s_j which are roots of

$$e^s - e^{-s} - sk(e^s + e^{-s}) = 0. \tag{4}$$

Hence

$$g(x, s_j) = A(s_j)e^{is_jz} + B(s_j)e^{-is_jz} = \int_{-1}^{1} e^{-s_jy}u(x, y)dy.$$

Now if $k>1$,[7] (4) has a real root at $s=0$ and an infinite sequence of conjugate imaginary roots. If $k<1$, (4) has three real roots and an infinite sequence of conjugate imaginary roots. As a matter of notation let $s_j=i\alpha_j$ ($i=(-1)^{1/2}$). If $u(x, y)$ is to be bounded for all x, s_j is real and hence all $A(s_j)$ and $B(s_j)$ are zero save for the coefficients A and B corresponding to the real s_j. If $u(x, y)$ is to be of finite exponential order, say $O(e^{\lambda z})$, then $|s_j| < \lambda$ and all $A(s_j)$ and $B(s_j)$ are zero for any $|s_j|$ greater than $|\lambda|$. Thus without going to the expense of solving a nonhomogeneous differential equation and evaluating complex integrals as Cooper did, we can get the same results he did by elementary methods.

We can of course obtain Cooper's expansion by noting that the set of functions $\{\sin \alpha_j y\}$ ($is_j=\alpha_j$) is a complete set of orthogonal functions over $-1 \leqq y \leqq 1$. If we then assume that $u(x, y)$ is of bounded variation with respect to y in $-1 \leqq y \leqq 1$ (or any other such condition which will insure an adequate representation) we have

$$\sum_{j=-\infty}^{\infty} C_j e^{i\alpha_j y} \int_{-1}^{1} e^{-i\alpha_j y'}u(x, y')dy' = u(x, y) \tag{5}$$

where the C_j's are the normalizing factors of the set $\{\sin \alpha_j y\}$. Then if $u(x, y)$ is bounded with respect to x, (5) appears as a sum of two terms (that is, the case $k>1$). If $u(x, y)$ is of finite exponential order with respect to x, (5) appears as a finite sum. It is to be noted that since no conditions other than order conditions were put on x, for large x, the A's and B's remain undetermined.

The technic we have employed can be applied to the solution of Laplace's equation in two or three variables, when the boundary of the domain over which we solve this equation is a rectangle. The same technic may be applied to the solution of characteristic value

[7] The critical case $k=1$, follows as the discussion in the text. In this case a first degree polynomial in x arises instead of the exponential functions.

problems associated with the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0$$

for various types of boundary conditions when the boundary is rectangular.

PURDUE UNIVERSITY

ON THE CONVERGENCE OF A CONTINUED FRACTION

T. F. GLASS AND WALTER LEIGHTON

It is known [1] that sufficient conditions for the convergence of the continued fraction

$$b_0 + \frac{a_1}{1 +} \frac{a_2}{1 +} \cdots, \tag{1}$$

where the elements are complex numbers, are

$$|a_2| \geqq 5, \quad |a_{2n}| \geqq 25/4, \quad |a_{2n-1}| \leqq 1/4, \quad n = 2, 3, 4, \cdots. \tag{2}$$

The purpose of this note is to extend this result.

THEOREM. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ $(n = 1, 2, 3, \cdots)$ *and if the numbers* $a_{2n} = \rho_{2n} e^{i\theta_{2n}}$ $(n = 1, 2, 3, \cdots)$ *satisfy the conditions*

$$\rho_{2n} \geqq 2(1 + r)^2[1 - \cos(\theta_{2n} + \theta_0)], \qquad 0 \leqq \theta_{2n} < \pi - \theta_0, \tag{3}$$

$$\rho_{2n} \geqq 4(1 + r)^2, \qquad \pi - \theta_0 \leqq \theta_{2n} \leqq \pi + \theta_0, \tag{4}$$

$$\rho_{2n} \geqq 2(1 + r)^2[1 - \cos(\theta_{2n} - \theta_0)], \qquad \pi + \theta_0 < \theta_{2n} \leqq 2\pi, \tag{5}$$

where $\theta_0 = 2 \arcsin r$, *the continued fraction* (1) *converges.*

To prove the theorem we employ the continued fraction

$$1 + \frac{x_1}{1 +} \frac{x_2}{1 +} \cdots \tag{6}$$

where

$$x_{2n} = \frac{(1 + a_{2n-1})(1 + a_{2n+1})}{a_{2n}}, \quad n = 2, 3, \cdots, \tag{7.1}$$

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$$x_{2n+1} = a_{2n+1}, \qquad n = 1, 2, \cdots, \tag{7.2}$$

$$x_1 = -a_1, \qquad x_2 = \frac{1 + a_2}{a_2}. \tag{7.3}$$

Since the convergence of (1) is independent of the choice of b_0, we set $b_0 = 1 - a_1$, and it follows [1] that the $2n$th convergent of (1) formally equals the $(2n+1)$st convergent of (6), while the $(2n+1)$st convergent of (1) is equal to the $2n$th convergent of (6). Thus (1) and (6) will converge or diverge together. By the first hypothesis of the theorem and (7.2) the numbers x_{2n+1} are bounded and lie in $\overline{R}$ the closed parabola $|z| - \Re(z) = 1/2$. It is thus sufficient to prove that the numbers x_{2n} defined in (7.1) and (7.3) subject to conditions (3), (4), and (5) are bounded and lie in the parabolic region described above [2].

To this end let

$$\begin{aligned} s_{2n+1} &= (1 + a_{2n-1})(1 + a_{2n+1}) = a_{2n}x_{2n} \\ &= r_{2n+1}e^{i\phi_{2n+1}}, \qquad n = 2, 3, \cdots, \end{aligned}$$

and set $x_{2n} = t_{2n}e^{i\omega_{2n}}$. It is clear that $t_{2n} \leqq \max r_{2n+1}/\min \rho_{2n} = (1+r)^2/\min \rho_{2n}$.

First suppose that a_{2n} lies in the region defined by (4). Then $\min \rho_{2n} = 4(1+r)^2$, $t_{2n} \leqq 1/4$ and x_{2n} will lie in $\overline{R}$. Next suppose that (3) holds, that is, that a_{2n} lies outside the cardioid $\rho = 2(1+r)^2[1 - \cos(\theta + \theta_0)]$ and in the angle $0 \leqq \theta < \pi - \theta_0$. Hence $\min \rho_{2n} = 2(1+r)^2[1 - \cos(\theta + \theta_0)]$ for each θ and thus

$$t_{2n} \leqq \frac{1}{2[1 - \cos(\theta_{2n} + \theta_0)]}, \quad 0 \leqq \theta_{2n} < \pi - \theta_0. \tag{8}$$

Further $\omega_{2n} = \phi_{2n+1} - \theta_{2n}$ from which it follows that $-\theta_0 - \theta_{2n} \leqq \omega_{2n} \leqq \theta_0 - \theta_{2n}$, and hence that $-\pi \leqq \omega_{2n} \leqq \theta_0$. It is clear that the right-hand member of (8) decreases steadily from $1/2(1 - \cos\theta_0)$ to $1/4$ as θ_{2n} increases from 0 to $\pi - \theta_0$ and at the same time ω_{2n} decreases steadily from ϕ_{2n+1} to $\phi_{2n+1} + \theta_0 - \pi$. The proof for the case when (3) holds may now be completed by proving that the point $(1/2\,(1 - \cos\theta_0), \theta_0)$ and the points

$$(1/2[1 - \cos(\theta + \theta_0)], -\theta - \theta_0), \qquad 0 \leqq \theta < \pi - \theta_0, \tag{9}$$

lie in $\overline{R}$, as an examination of a simple figure will show. The first point evidently lies on the parabola since the equation of the parabola in polar coordinates is

$$r = \frac{1}{2(1 - \cos\theta)}. \tag{10}$$

The substitution $\lambda = -\theta - \theta_0$ proves immediately that the points (9) lie on (10).

By symmetry it is clear that when a_{2n} lies in the region defined by (5) the corresponding complex number x_{2n} lies in $\overline{R}$. This completes the proof of the theorem.

COROLLARY 1. *If* $|a_{2n+1}| \leqq r \leqq 1/4$ *and* $|a_{2n}| \geqq 4(1+r)^2$, *the continued fraction* (1) *converges.*

It is clear that an analogous theorem to the above may be proved with the roles of the even and the odd elements interchanged.

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2. W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 155–172.

THE MARION INSTITUTE AND
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EXPANSIONS OF QUADRATIC FORMS

RUFUS OLDENBURGER

1. **The problem.** A quadratic form Q with coefficients in a field K, whose characteristic is different from 2, is usually given as a linear combination

$$\sum_{i=1}^{n} a_{ij}x_ix_j \tag{1}$$

of products $\{x_i\,x_j\}$, where (a_{ij}) is symmetric. The sum (1) is one of the type

$$\sum_{i=1}^{r} L_iM_i, \tag{2}$$

where the L's and M's are linear forms. In general the decomposition (1) is not the most economical way of writing Q as a sum of the type (2) in the sense that r is a minimum for Q. In treating algebras associated with quadratic forms E. Witt[1] showed that the form Q is equivalent under a nonsingular linear transformation to a decomposition

$$\sum_{i=1}^{\sigma} y_iz_i + \sum_{i=1}^{r-2\sigma} v_iu_i^2, \tag{3}$$

where the last sum is a nonzero form, and r is the rank of Q. In the present paper we shall show that the minimum τ for Q is $r-\sigma$. Thus this minimum τ is determined by the rank r and the "characteristic" σ of Q. This characteristic[2] is the maximum number σ of linearly independent linear forms $L_1, \cdots, L_\sigma$ such that the rank of $Q+\lambda_1L_1^2+\cdots+\lambda_\sigma L_\sigma^2$ is the same as the rank of Q for all values of the λ's. The form Q has characteristic σ *if and only if* Q has the *canonical splitting* $G+H$, where G has characteristic σ and rank 2σ, while H has characteristic 0 and rank $r-2\sigma$. The form G has a decomposition (2) with $\tau=\sigma$. The decomposition (3) is one such that the first sum is a form G of the type described and the other a form H. Thus it will be proved

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[1] E. Witt, *Theorie der Quadratischen Formen in beliebigen Körpern*, J. Reine Angew. Math. vol. 176 (1937) p. 35.

[2] Rufus Oldenburger, *The index of a quadratic form for an arbitrary field*, Bull. Amer. Math. Soc. abstract 48-5-162.

that the decomposition (3) corresponding to a canonical splitting $G+H$ of Q is one with a minimum number of terms.

Like the rank of Q the characteristic of Q has the property that this characteristic changes at most by 1 under addition of a term λL^2, L linear, to Q. We shall prove here that actually the minimum τ, the characteristic σ defined above, and the index[3] (if K is real) possess this property of changing at most by 1 under additions of the type LM to Q, where L and M are arbitrary linear forms.

We recall that the rank r of Q is the minimum τ for which Q can be written as a sum (2), where for each i the forms L_i and M_i are linearly dependent. Thus both the rank alone, and the rank and characteristic of Q, yield minimum properties of expansions of Q invariant under nonsingular linear transformations on the variables in Q.

It will be understood throughout the present paper that the coefficients are in a field K of the type specified above. The field K is otherwise unrestricted, except where K is taken to be the real or complex fields.

2. **Solution of the minimum problem.** The following lemma needs no proof.

LEMMA 1. *The characteristic of a quadratic form Q is invariant under nonsingular linear transformations on Q.*

The lemma to follow was proved elsewhere.[4]

LEMMA 2. *The characteristic of a quadratic form Q changes at most by 1 under addition to Q of a term λL^2, L linear and λ in the given field.*

LEMMA 3. *The characteristic of a quadratic form Q is at least as great as the characteristic of each form Q^* obtained from Q by imposing homogeneous linear relations on the variables in Q.*

We write Q as in (1), and suppose that Q^* is obtained from Q by equating $x_1, \cdots, x_{e-1}$ to zero for some e. The matrix $A=(a_{ij})$ of Q can be written as

$$\begin{Vmatrix} * & * \\ * & B \end{Vmatrix},$$

where B is the matrix $(a_{ij})\,[i, j=e, \cdots, n]$ of Q^*, and the asterisks in A indicate minors of A. We let α designate the characteristic of

[3] The *index* of a real quadratic form Q is the number k of $+$ signs in a canonical form $x_1^2+\cdots+x_k^2-x_{k+1}^2-\cdots-x_r^2$ to which Q is equivalent.

[4] See the above reference to a paper by R. Oldenburger.

the form Q^*. Since Q^* has a canonical splitting, as described in §1, the form Q^* is equivalent under a nonsingular transformation to the form

$$\sum_{i=1}^{\alpha} u_i v_i + F, \tag{4}$$

where F is a form of characteristic 0 whose variables are linearly independent of the u's and v's. With the form (4) we may associate the symmetric matrix

$$\left\| \begin{matrix} 0 & 0 \\ 0 & C \end{matrix} \right\|$$

of order $n-e+1$, where C is a nonsingular minor of the type

$$\left\| \begin{matrix} 0 & 0 & I \\ 0 & D & 0 \\ I & 0 & 0 \end{matrix} \right\|,$$

the minor I being an identity matrix of order α [arising from the summation in (4)]. It follows from elementary matrix considerations that there is a nonsingular matrix N such that

$$N'AN = \left\| \begin{matrix} E & 0 \\ 0 & 0 \end{matrix} \right\|,$$

where E is a nonsingular minor with the shape

$$E = \left\| \begin{matrix} * & * & * & I \\ * & * & * & 0 \\ * & * & D & 0 \\ I & 0 & 0 & 0 \end{matrix} \right\|,$$

and N' designates the transpose of N. We let $T = T(y_1, \cdots, y_n)$ be a quadratic form in $y_1, \cdots, y_n$ associated in the usual manner with $N'AN$. The rank of $T+\lambda_1 y_1^2+ \cdots +\lambda_\alpha y_\alpha^2$ is the same as the rank of T for all values of the λ's. Since T is equivalent to Q the characteristic of Q is at least α.

Suppose now that we impose homogeneous linear relations $Z_1=0, \cdots, Z_s=0$ on the variables $x_1, \cdots, x_n$ in Q. It is no restriction to take these Z's to be linearly independent forms. We may therefore use these Z's and enough of the x's to obtain a set of linearly independent forms, which we may employ as the n variables in terms of which Q is expressed. By Lemma 1 this change of vari-

ables leaves the characteristic invariant. Thus the problem which arises when the Z's are set equal to zero reverts to the above case where $x_1 = \cdots = x_{\epsilon-1} = 0$.

THEOREM 1. *The minimum τ for which a quadratic form Q with rank r and characteristic σ has the expansion* (2), *where the L's and M's are linear forms, is $r-\sigma$.*

We suppose that Q is written as (2), where τ is a minimum. If for some i and element k_i we have $M_i \equiv k_i L_i$, we write $k_i L_i^2$ in place of $L_i M_i$ (i not summed). Thus we can split the sum (2) into $R+S$, where

$$R = \sum_{i=1}^{s} L_i M_i, \qquad S = \sum_{i=1}^{t} \nu_i N_i^2, \tag{5}$$

L_i being linearly independent of M_i for each i, and the N's being linear forms. The L's form a set of linearly independent linear forms, since otherwise we can write R as a sum of products of linear forms with less terms. As in §1 we write a canonical splitting of Q as $G+H$. Since Q has rank r, we may take Q to be a form in r independent variables. Since the rank of S is t, we have $t \geqq r-2s$. If $s<\sigma$, we have $s+t>r-\sigma$, whence the decomposition corresponding to the canonical splitting $G+H$ has less terms than (2). Thus $s \geqq \sigma$, and we can write $s=\sigma+\rho$ for a $\rho \geqq 0$.

We relabel the subscripts on the L's, M's, and N's if necessary so that the forms in the set Σ, where

$$\Sigma = (L_1, \cdots, L_{\sigma+\rho}, M_1, \cdots, M_\xi, N_1, \cdots, N_\xi),$$

yield a minimal basis for the L's, M's and N's. Here $\xi \geqq r-\sigma-\rho-\zeta$. If (2) is a more economical decomposition than that which arises from $G+H$, we have $t \leqq r-\rho-2\sigma-1$. Now $t \geqq r-\sigma-\rho-\zeta$. These inequalities yield $\zeta \geqq \sigma+1$. We suppose that ζ satisfies this inequality. We let Q' designate the form

$$Q - \sum_{i=\xi+1}^{t} \nu_i N_i^2.$$

Since by Lemma 2 the characteristic changes at most by 1 under each subtraction with $\nu_i N_i^2$ (i not summed), the index α of Q' is such that

$$\alpha \leqq \sigma + t - \xi.$$

Eliminating t with the aid of an inequality relation above, we have $\alpha \leqq \zeta - 1$.

We take the linear forms in the set Σ to be the variables in terms of which the form Q' above is expressed. Setting $L_{\zeta+1}, \cdots, L_{\sigma+\rho}=0$, we obtain from Q' a form Q'' with index ζ. By Lemma 3, we have $\alpha \geqq \zeta$, giving us a contradiction. It follows that $\tau=r-\sigma$.

For the complex field the characteristic σ of Q is $[r/2]$, whereas for the real field σ is the minimum of the indices of Q and $-Q$. These results yield Corollary 1.

COROLLARY 1. *For the complex field the minimum number τ of Theorem* 1 *is $r-[r/2]$. For the real field τ is the maximum of the indices of Q and $-Q$.*

Witt proved[5] that a form is a zero form if and only if the characteristic σ of this form is greater than 0.

COROLLARY 2. *The form Q of Theorem* 1 *is a zero form if and only if $\tau \neq r$.*

By Theorem 1 the sum (2), where τ is a minimum, is a sum with the R and S of (5) satisfying $R=G$, $S=H$, the sum $G+H$ being a canonical splitting of Q.

Although addition of a term LM, L and M linear, may change the rank r of Q by 2, this is not true of the index σ and $\tau=r-\sigma$ as we shall now prove.

THEOREM 2. *Under addition of a term LM, L and M linear, to a quadratic form Q, the characteristic σ, and the minimum number τ for decompositions of type* (2), *change at most by* 1.

We write Q as a sum (2) where τ takes on the minimum value $r-\sigma$, the rank of Q being r. We let τ', r', σ' designate the analogues for $Q'=Q+LM$ of τ, r, σ for Q. Since

$$Q' = \sum_{i=1}^{\tau} L_i M_i + LM,$$

we clearly have $\tau' \leqq \tau+1$. Thus τ changes at most by 1 under the addition of LM to Q.

We suppose that L and M are linearly independent of each other and of the variables in Q so that $r'=r+2$. We write Q as $R+S$, where R and S are given by (5) with $s=\sigma$, $t=r-\sigma$, whence

$$Q \equiv \sum_{i=1}^{\sigma} L_i M_i + \sum_{i=1}^{r-\sigma} \nu_i N_i^2. \tag{6}$$

[5] See the above reference to a paper by E. Witt

The form R', where $R'=R+LM$, has index $\sigma+1$ and rank $2(\sigma+1)$, from which it follows that Q' has the canonical splitting $R'+S'$ with $S'\equiv S$. Thus Q' has index $\sigma+1$.

If L and M are taken linearly dependent, or one or both of the forms L, M are restricted to be linear forms in the variables of Q, by Lemma 3 we obtain from the form Q' of the preceding paragraph a form Q^* whose characteristic does not exceed $\sigma+1$. Thus in any case $\sigma'\leqq\sigma+1$, whence also $\sigma\leqq\sigma'+1$.

If the rank of $Q+LM$ is less than the rank of Q, Theorem 2 implies that the characteristic of $Q+LM$ does not exceed that of Q, whereas if the addition of LM to Q decreases the rank of Q by 2, this addition also decreases the characteristic of Q.

We have the following analogue of Theorem 2.

THEOREM 3. *Under addition of a term LM, L and M linear, to a real quadratic form Q, the index of Q changes at most by* 1.

We write Q as the sum $P+N$ of a positive definite form N and a negative definite form N, the rank of Q being the sum of the ranks of P and N. We suppose that Q is written in any way as a sum $P'+N'$, where P' and N' are positive definite and negative definite forms respectively. The index α of Q is the rank of P. We let β designate the rank of P', whence

$$P' = \sum_{i=1}^{\beta} p_i P_i^2$$

for linear forms $P_1, \cdots, P_\beta$. We suppose that $\alpha>\beta$. Setting $P_1=\cdots=P_\beta=0$, we have $P\not\equiv 0$, while $P+N$ is negative definite, a contradiction. Thus $\alpha\leqq\beta$.

The form LM can be written as the difference R^2-S^2, where R and S are linear forms, or one of the terms R, S is zero. The form $Q+LM$ is a sum of the positive definite form $P+R^2$, and negative definite part $N-S^2$. Since the rank of $P+R^2$ differs at most by 1 from the rank of P, the index of $Q+LM$ does not exceed $\alpha+1$. It follows that the indices of Q and $Q+LM$ differ at most by 1.

ILLINOIS INSTITUTE OF TECHNOLOGY

SECTIONS OF CONTINUOUS COLLECTIONS

J. H. ROBERTS AND PAUL CIVIN

In the present note we establish the following

THEOREM. *Suppose G is a continuous collection*[1] *of closed and compact sets filling a separable metric space X. Suppose further that the space G, considered as a decomposition space, has dimension at most n. Then there is a closed subset K of X, such that for each $g \in G$, the set $g \cdot K$ is nonvacuous and consists of at most $(n+1)$ points.*

We call such a point set K an $(n+1)$-section of the collection G. Thus a 1-section of G is a true section. G. T. Whyburn[2] has shown that if the elements of G are 0-dimensional and G is a dendrite, then G admits a true section. The present result gives only a 2-section, but there is no hypothesis on the dimension of the elements of G. For $n=1$, it is known that in general G does not admit a true section. For $n>1$ it is not known whether the present result gives the best possible constant.

We first establish the theorem in the 0-dimensional case.

LEMMA. *Suppose G is 0-dimensional, and ϵ is a given positive number. Suppose W is an open set in X such that $W \cdot g \neq 0$ for each $g \in G$. Then there is an open set E in X such that $\overline{E} \subset W$, $E \cdot g \neq 0$ for every $g \in G$, and the diameter of $E \cdot g < \epsilon$ for each $g \in G$.*

Let $f(x)$ be a homeomorphism of M, a subset of the Cantor set, into G.[3] In the product space $M \times X$, consider the set A of points (x, y) with $x \in M$ and $y \in f(x)$. For $x \in X$ there is a unique $y = y(x)$ in M such that $x \in f(y)$. The function $t(x) = (y(x), x)$ is a homeomorphism of X into A.

In the space A, the open set $t(x)$ and the continuous collection H of elements $t(g)$ for $g \in G$ satisfy the properties of W and G stated in the hypothesis of the lemma. Furthermore, the diameter of a set Z in A is not smaller than the diameter of $t^{-1}(Z)$. Hence all we need show is that there exists an open set E satisfying the theorem relative to the open set $t(W) = U$ and the continuous collection H.

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[1] *A continuous collection filling a space X*, is a collection G of sets g such that: (1) If $x \in X$, then $x \in g$ for exactly one g. (2) If $x \in g$, $x_n \in g_n$ and $x_n \to x$, then $\lim g_n = g$.

[2] *A theorem on interior transformations*, Bull. Amer. Math. Soc. vol. 44 (1938) pp. 414–416.

[3] P. Urysohn, *Sur les multiplicités Cantoriennes*, Fund. Math. vol. 7 (1926) p. 77.

For each $p \in U$ there is an open set U_p such that (1) $U_p \supset p$, (2) $\overline{U}_p \subset U$, (3) the diameter of $U_p < \epsilon$, and (4) the projection[4] V_p of U_p upon M is both open and closed. The collection $\{V_p\}$ is an open covering of M and therefore there is a countable subcollection $\{V_{p_i}\}$ covering M. The collection of sets $\{W_i\}$ where W_i is defined by the relations

$$W_1 = V_{p_1}, \qquad W_i = V_{p_i} - \sum_{j=1}^{i-1} V_{p_j}$$

is a covering of M by mutually exclusive open sets. Let Y_i denote the open subset of U_{p_i} whose projection is W_i and let $E = \sum_{i=1}^{\infty} Y_i$. The open set E has the required properties relative to the space A, the open set U, and the continuous collection H.

In order to prove that $\overline{E} \subset U$ it is sufficient to show that $\overline{E} = \sum_{i=1}^{\infty} \overline{Y}_i$. Suppose $p \in \overline{E}$ and $p \notin \sum_{i=1}^{\infty} \overline{Y}_i$. Then there is a sequence $p_n \to p$ and $p_n \in Y_{i_n}$. Suppose $p \in g$, and $\pi(g) \subset W_j$, where π denotes the projection of A on M. Since $p_n \to p$, $\pi(p_n) \to \pi(p)$. But $\pi(p_n) \not\subset W_j$ for more than a finite number of n. This contradicts the fact that W_j is open.

Now E intersects each g since the sequence $\{W_i\}$ is a covering of M. Also, since the sets W_i are mutually exclusive, if $Y_i \cdot g \neq 0$ then $Y_j \cdot g = 0$ for $i \neq j$. Then, as $Y_i \subset U_{p_i}$ and the diameter of $U_{p_i} < \epsilon$, the diameter of $E \cdot g < \epsilon$ for every $g \in H$. This proves the lemma.

The theorem for the 0-dimensional case follows by considering a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ and a sequence of open sets $\{E_n\}$ such that $\overline{E}_{n+1} \subset E_n$, $E_n \cdot g \neq 0$ for $g \in G$, and the diameter of $E_n \cdot g < \epsilon_n$, for $g \in G$. The common part K of the sets E_n is closed. For $g \in G$, the set $K \cdot g$ consists of exactly one point, since g is compact and $\epsilon_n \to 0$.

The theorem for the n-dimensional case follows by considering an at most $(n+1)$-to-one closed mapping $f(x)$ of a subset M of the Cantor set[5] into G. In the product space $M \times X$ consider the set A of points (y, x) with $y \in M$ and $x \in f(y)$. The sets (y, x) for y fixed and $x \in f(y)$ form a 0-dimensional continuous collection H which fills A. The mapping $t(y, x) = x$ is a closed, at most $(n+1)$-to-one mapping of A into X. By the theorem for the 0-dimensional case, there is a true section K of the collection H in the space A. The set $t(K)$ gives the required $(n+1)$-section of the continuous collection G.

DUKE UNIVERSITY

[4] That is, V_p is the set of $x \in M$ such that $(x, y) \in U_p$.

[5] See J. H. Roberts, *A theorem on dimension*, Duke Math. J. vol. 8 (1941) p. 572, Theorem 9.1. The mapping ϕ_n as actually defined is a closed mapping, although this result is not specifically stated in the theorem.

FUNCTIONAL TOPOLOGY

MARSTON MORSE

Introduction. The present paper gives the fundamental existence proof for a homotopic critical point under hypotheses which are less stringent than those employed in the author's recent fascicule on functional topology and abstract variational theory [2]. This relaxation of hypotheses seems to be necessary if one is to extend the variational theory in the large to non-regular problems. Non-regular integrals include some of the most important and interesting integrals such as the Jacobi least action integral[1] in the three body problem of celestial mechanics. The author believes that a topological basis for the planetary orbits will be disclosed by studying the contour manifolds of this Jacobi integral.

The results of this paper will be used to extend the theory in the large to non-regular problems in two papers by Morse and Ewing.

1. **The metric spaces** M, L, J. We are concerned with a compact metric space M of elements $p, q, r, \cdots$ and distance $pq, pr, \cdots$. We shall deal with two functions $J(p)$ and $L(p)$, bounded, single-valued, and lower semi-continuous on M. In the applications p will be a curve joining two fixed points in some space, the distance pq will be the Fréchet distance between curves, while $J(p)$ and $L(p)$ will be integrals along p, with L the length of p.

Beside the metric M we shall use two other metrics, an L-metric with a distance

$$|pq| = pq + |L(p) - L(q)|,$$

and a J-metric with a distance,

$$pq + |J(p) - J(q)|.$$

We shall refer to the corresponding spaces as the spaces L and J. We shall make the following hypothesis.

HYPOTHESIS. *Convergence on L to a point p shall imply convergence on J to p.*

We shall not assume that convergence on J to p implies convergence on L to p. Convergence on L or J clearly implies convergence on M. Terms such as neighborhood, compact, and so on

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[1] For other examples see paper [1] by McShane.

will be preceded by the letters M, L, or J according to which metric is used to define them. Every M-neighborhood of a point p contains an L-neighborhood of p, but not conversely. Every J-neighborhood of p contains an L-neighborhood of p, but not conversely.

The subset of points on which $J \leqq c$ will be denoted by J^c. The set J^c is M-compact since J is lower semi-continuous on M. But J^c will not in general be L or J-compact. In fact an M-compact subset A would be L-compact if and only if $L(p)$ were M-continuous on A. Similarly with J-compactness. In seeking to cover sets such as J^c with a finite number of neighborhoods we accordingly use M-neighborhoods, although it would be simpler if we could use L-neighborhoods.

On the other hand the deformations which we shall use in later papers are adequate only if their continuity is L-continuity. We shall thus be using L-continuous deformations defined over M-neighborhoods. But J-neighborhoods enter also, since there are important properties which can be established for J-neighborhoods but not for L or M-neighborhoods.

Our chains and cycles shall be defined on L using L-continuity. They shall be finite singular chains and cycles, taken mod 2, see [3, p. 146]. The point set bearing a singular chain is L-compact and hence J-compact. We shall term the least upper bound of J on an arbitrary set E, the *J-height* of E. In general we shall say that a point p is *above* or *below* q according as $J(p)$ is greater or less than $J(q)$. We shall admit *relative* k-cycles u in which the modulus is always a set J^c in which c is less than the J-height of u.

2. **J-deformations.** Let E be a subset of L. Let I be an interval $0 \leqq \tau \leqq a$ $(a > 0)$. By $E \times I$ we shall mean the product of E and I, assigning the usual metric to the space $E \times I$. We shall admit deformations D of E which replace a point p found on E at the time $\tau = 0$ by a point $q(p, \tau)$ on L at the time τ $(0 \leqq \tau \leqq a)$. If D is to be admissible we require:

(a) *That $q(p, \tau)$ map $E \times I$ continuously into L.*

(b) *That for p fixed $q(p, \tau)$ map I continuously into M uniformly with respect to (p, τ) on $E \times I$.*

We shall say that D is a *weak J-deformation*, if for p on E and for $q = q(p, \tau)$, $J(p) - J(q) \geqq 0$ for each τ on I. A weak J-formation $q(p, \tau)$ will be said to be *proper* on E, if when τ is bounded from 0 on I,

$$J(p) - J(q) > e > 0, \qquad q = q(p, \tau),$$

where e is a constant independent of p on E.

A point p will be said to be *homotopically J-ordinary* if some J-neighborhood of p admits a proper J-deformation. A point which is not homotopically J-ordinary will be termed *homotopically J-critical.*

We shall say that J is *upper-reducible* at p if for each constant $a > J(p)$ there exists a weak J-deformation of some M-neighborhood of p which is proper for points initially above a.

The reader may wonder why a J-neighborhood is used in the above definition of a J-critical point while an M-neighborhood is used in the definition of upper-reducibility. A J-neighborhood is used in this connection because otherwise we are unable to prove that a rectifiable curve which is not an extremal is homotopically J-ordinary.[2] This reflects the fact that being an extremal is a consequence of properties possessed by weak neighborhoods such as J-neighborhoods. On the other hand an M-neighborhood is used in the definition of upper-reducibility since otherwise we could not prove the fundamental Theorem 3.1 of this paper. For, as one sees from its definition, upper-reducibility is not concerned merely with points q at which $J(q)$ is near $J(p)$, so that a J-neighborhood would be too restrictive.

We come to products of deformations. Let D be a deformation of a set A. The set of final images of points of A under D will be denoted by $D(A)$. Let $B_1, \cdots, B_n$ be a set of weak J-deformations such that B_1 is applicable to A, B_2 to $B_1(A)$, and more generally B_{i+1} is applicable to $B_i \cdots B_1(A)$. In such circumstances the deformations $B_n \cdots B_1$ will be said to define a product deformation Δ. The deformation Δ is applicable to A. One sees that Δ is a weak J-deformation of A.

The following lemma shows how a weak J-deformation can be extended as a weak J-deformation beyond a local domain of definition.

LEMMA 2.1. *Let B be a subset of L and s a point of B. Let S_r be the intersection with B of a spherical M-neighborhood of s of radius r. Suppose that S_{3e} admits a weak J-deformation D. Then D can be replaced by a weak J-deformation θ of B such that $\theta \equiv D$ for points initially on S_e and θ is the null deformation for points initially exterior to S_{2e}.*

Suppose the time τ in D varies on the interval $(0, a)$. Under θ, τ shall likewise vary on $(0, a)$. Points of B initially on S_e shall be deformed under θ as under D while points of B initially exterior to S_{2e} shall be held fast. Points p of B at a distance ps from s such that

$$e \leqq ps \leqq 2e, \tag{2.1}$$

[2] This theorem is the basic generalization of the Euler theorem that a curve (of class C') which is not an extremal is not minimizing.

shall be deformed under θ as follows. Let $t(p)$ divide the interval $(0, a)$ in the ratio inverse to that in which ps divides the interval (2.1). That is, let

$$\frac{t(p)}{a} = \frac{2e - ps}{e}, \qquad e \leqq ps \leqq 2e.$$

Under θ points p of B which satisfy (2.1) initially shall be deformed as under D until τ reaches $t(p)$ and shall be fast thereafter.

Let $q(p, \tau)$ be the image of p under θ. Recall that ps, and hence $t(p)$, vary continuously as p varies L-continuously on $\overline{S_{2e}-S_e}$. It will be convenient to set $t(p) \equiv a$ for p on S_e and $t(p) \equiv 0$ for p on $B - S_{2e}$. Then $t(p)$ is defined and continuous as p varies on B. To establish the continuity of $q(p, \tau)$ one breaks the domain of the pairs (p, τ) into the two domains

$$[0 \leqq \tau \leqq t(p)] \qquad [p \text{ on } B], \tag{2.3}$$

$$[t(p) \leqq \tau \leqq a] \qquad [p \text{ on } B], \tag{2.4}$$

with the set on which $\tau = t(p)$ in common. On the second domain $q(p, \tau)$ is constant. On the first domain $q(p, \tau)$ equals the point function defining D. The functions $q(p, \tau)$ defined over these separate domains obviously combine to define a function $q(p, \tau)$ with the properties of a weak J-deformation of B.

3. **The fundamental theorem.** Two different sets of neighborhoods U and V, respectively, covering a given space, will be said to be *equivalent* if each neighborhood U of each point p contains a neighborhood V of p, and conversely. In this sense the set of J-neighborhoods is equivalent to the set of neighborhoods $V(p)$ defined by conditions of the form

$$pq < \delta, \qquad J(q) < J(p) + e, \tag{3.1}$$

where p is fixed and δ and e are arbitrary positive constants. This is a consequence of the lower semi-continuity of $J(p)$ for p on M. For the relation $pq < \delta$ implies the relation

$$J(p) - e < J(q) \tag{3.2}$$

if δ is sufficiently small. Conditions (3.1) and (3.2) taken jointly define a set of neighborhoods equivalent to the set of J-neighborhoods, so that the set of neighborhoods (3.1) is equivalent to the set of J-neighborhoods. We shall therefore feel free to replace J-neighborhoods by neighborhoods of the type (3.1).

We continue with a fundamental theorem.

THEOREM 3.1. *Let C be an M-compact set with J-height c, containing no homotopic J-critical points at the level c. If J is upper-reducible at each point of C there exists a weak J-deformation of a J-neighborhood of C into a set with J-height less than c.*

I. If p is a point of C at the level c, p is homotopically J-ordinary. There accordingly exists a spherical M-neighborhood $V(p)$ of p and a positive constant $c(p)>c$ such that the intersection $J^{c(p)}\cdot V(p)$ admits a proper J-deformation $D(p)$.

II. If p is a point of C below c the upper-reducibility of J at p implies the following: there exists a spherical M-neighborhood $V(p)$ of p and a constant $a(p)<c$ such that $V(p)$ admits a weak J-deformation $D(p)$ which is a proper deformation of the subset of $V(p)$ above $a(p)$.

Let $U(p)$ and $R(p)$ be spherical M-neighborhoods of p with radii one-third and one-sixth that of $V(p)$, respectively. Since C is M-compact there exists a finite set of neighborhoods $R(p)$, say neighborhoods $R_1, \cdots, R_n$ of points $p_1, \cdots, p_n$, respectively, covering C. For points p_i at the level c, set $\mu=\min c(p_i)$. For points p_i below c set $\nu=\max a(p_i)$. We shall apply Lemma 2.1 setting $s=p_i$, $B=J^\mu$ and e is equal to the radius of $U_i=U(p_i)$. With this choice of s, B and e we infer the existence of a weak J-deformation H_i of J^μ which deforms $U_i\cdot J^\mu$ as does $D(p_i)$.

Let r equal the minimum of the radii r_i of the spheres R_i. Let θ_i be a weak J-deformation defined by taking the time interval in H_i so short that the maximum M-displacement under θ_i is less than r/n. The product deformation $\Delta=\theta_n\cdots\theta_1$ is a weak J-deformation of J^μ and displaces no point an M-distance in excess of $r\leqq r_i$. In particular we can conclude that

$$\theta_{i-1}\cdots\theta_1(R_i\cdot J^\mu)\subset U_i\cdot J^\mu. \tag{3.3}$$

Let S_i be the set defined by the left number of (3.3) and let C_i be the subset of S_i above ν. The deformation θ_i is proper on C_i since $\theta_i\equiv D(p_i)$ on C_i. If $C_i\neq 0$, the J-height of C_i is accordingly diminished under θ_i by a positive constant η_i. If $C_i=0$, set $\eta_i=0$. Let 2η be a positive constant less than each $\eta_i\neq 0$ and such that $c+\eta<\mu$ (recall that $c<\mu$). Then Δ deforms the set

$$(R_1+\cdots+R_n)\cdot J^{c+\eta} \tag{3.4}$$

into a set with a J-height at most $\max(\nu, c-\eta)<c$.

The set (3.4) contains a J-neighborhood of C and the proof of the theorem is complete.

A class K of k-cycles on L, either relative or absolute, will be said to form a *J-class* if any cycle into which a cycle u of K can be carried by a weak J-deformation is also in K. With this understood we have the following corollary of the theorem.

COROLLARY 3.1. *Suppose J is upper-reducible at each point of an M-compact set C. If there exists a J-class K of k-cycles such that K includes a cycle in each J-neighborhood of C but no cycle with a J-height less than that of C there exists at least one homotopic J-critical point with the J-height of C.*

In applying this corollary one is led to the two following specializations.

COROLLARY 3.2. *Let K be a homology class of absolute k-cycles, non-bounding on L, and let c be the greatest lower bound of J-heights of cycles of K. If J is upper-reducible at each point of J^c there exists a homotopic J-critical point of J at the level c.*

Corollary 3.2 follows from Corollary 3.1 upon taking C as J^c.

COROLLARY 3.3. *Let K be a homology class composed of k-cycles* mod J^a, *non-bounding* mod J^a *on L, and let c be the greatest lower bound of J-heights of cycles of K. If $c>a$ and if J is upper-reducible at each point of J^c there exists a homotopic J-critical point at the level c.*

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RECURRENCE FORMULAS FOR CERTAIN DIVISOR FUNCTIONS

D. H. LEHMER

Numerical functions giving the excess of the number of divisors of n of one sort over the number of divisors of a second sort were introduced over a century ago from the theory of binary quadratic forms and from the theory of elliptic functions. Later a systematic discussion of several of these functions was made by J. W. L. Glaisher, who, by means of theta function identities, found recurrences with gaps for these functions. The purpose of this note is to point out that these functions are sums over divisors of certain periodic Lucas functions and that sums of Lucas functions in general satisfy triangular number recurrence relations as given by (6) below. This formula is specialized in several ways, first to periodic Lucas functions, then to degenerate functions, and then to the more typical cases related to the Pell equation and Fibonacci's series.

Formula (6) may be obtained in the following elementary manner from the famous triple product identity of Gauss and Jacobi.[1]

$$(1) \qquad \prod_{r=1}^{\infty} \{(1 - x^{2r})(1 + ax^{2r-1})(1 + a^{-1}x^{2r-1})\} = \sum_{n=-\infty}^{\infty} a^n x^n .$$

If in (1) we set $x = t^{1/2}$ and $a = -\alpha t^{1/2}$ we obtain at once

$$(2) \quad (1 - \alpha)\prod_{r=1}^{\infty} \{(1 - t^r)(1 - \alpha t^r)(1 - \alpha^{-1}t^r)\} = \sum_{n=-\infty}^{\infty} (-\alpha)^{n+1} t^{n(n+1)/2}.$$

If now we introduce θ by $\alpha = e^{2\theta i}$ and write

$$(3) \qquad P(\theta, t) = \prod_{r=1}^{\infty} \{(1 - t^r)(1 - 2t^r \cos 2\theta + t^{2r})\},$$

then, on combining the terms on the right of (2) for $n = r - 1$ and $-r$,

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[1] K. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Königsberg, 1829, §64; *Gesammelte Werke*, Berlin, 1881, vol. 1 pp. 232–234. Other elementary proofs of (1) (not depending on elliptic functions) have been given by A. Cauchy, C. R. Acad. Sci. Paris vol. 17 (1843) pp. 523–531; *Oeuvres*, vol. 8, pp. 42–50; A. Enneper, *Elliptische Functionen*, Halle, 1876, pp. 74–77; J. Tannery and J. Molk, *Éléments de la theorie des fonctions elliptiques*, Paris, 1896, vol. 2 pp. 10–13; G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1938, pp. 280–281.

we can write (2) in the form

$$P(\theta, t) = \sum_{r=1}^{\infty} t^{r(r-1)/2}(-1)^{r+1} \frac{\sin (2r-1)\theta}{\sin \theta}. \tag{4}$$

Differentiating (3) and (4) with respect to θ we find, on substituting for $P(\theta, t)$ from (4):

$$\left\{\sum_{k=1}^{\infty} (-1)^k t^{k(k-1)/2} \frac{\sin (2k-1)\theta}{\sin \theta}\right\}\left\{\sum_{r=1}^{\infty} \frac{t^r}{1-2t^r \cos 2\theta + t^{2r}}\right\}$$
$$= (1/4) \csc^2 \theta \sum_{k=1}^{\infty} (-1)^k t^{k(k-1)/2}\left\{\frac{k \cos (2k-1)\theta}{\cos \theta} - \frac{\sin 2k\theta}{\sin 2\theta}\right\}. \tag{5}$$

Let us consider the general Lucas function

$$U_n = \frac{a^n - b^n}{a-b}$$

where a, b are the roots of $x^2 - Px + Q = 0$. Supposing that $Q \neq 0$ we define q by

$$q = Q^{-1/2}$$

where a definite square root of Q is taken. Finally we introduce the divisor sum D_n by

$$D_n = \sum_{\delta | n} U_\delta q^\delta$$

where δ ranges over the positive divisors of n. Let the parameter θ in (5) be taken so that $2 \cos 2\theta = Pq$, and hence

$$\frac{\sin 2n\theta}{\sin 2\theta} = q^{n-1} U_n.$$

Then, since

$$U_n = PU_{n-1} - QU_{n-2},$$

we have

$$\begin{aligned} U_n q^n &= PqU_{n-1}q^{n-1} - q^2 QU_{n-2}q^{n-2} \\ &= 2 \cos 2\theta U_{n-1}q^{n-1} - U_{n-2}q^{n-2}. \end{aligned}$$

Therefore

$$(1 - 2x \cos 2\theta + x^2) \sum_{m=1}^{\infty} x^m q^m U_m = xq.$$

Setting $x = t^r$ and summing over $r = 1, 2, 3, \cdots$, we get

$$q\sum_{r=1}^{\infty}\frac{t^r}{1-2t^r\cos 2\theta+t^{2r}}=\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}t^{rm}q^mU_m=\sum_{n=1}^{\infty}D_nt^n.$$

Substituting this into (5) and comparing coefficients of t^n on both sides of the resulting equation we find

$$D_n-D_{n-1}\frac{\sin 3\theta}{\sin\theta}+D_{n-3}\frac{\sin 5\theta}{\sin\theta}-D_{n-6}\frac{\sin 7\theta}{\sin\theta}+-\cdots$$

$$(6)\qquad =\begin{cases}(q/4)\csc^2\theta(-1)^{k+1}\left\{\dfrac{k\cos(2k-1)\theta}{\cos\theta}-\dfrac{\sin 2k\theta}{\sin 2\theta}\right\} & \text{if } n=k(k-1)/2\\ 0 & \text{otherwise.}\end{cases}$$

Here the left side involves only such D's as have subscripts differing from n by triangular numbers.

Of the special cases of (6) perhaps the most interesting are those in which $q=1$ and θ is a simple rational multiple of π. In these cases the function D_n belongs to a general class which is composed of functions of which the typical one may be denoted by

$$E_m(n)=E_m(n\,|\,\alpha_1,\alpha_2,\cdots;\beta_1,\beta_2,\cdots)$$

and is defined as the excess of the number of those divisors of n of one of the forms $mx+\alpha_i$ over the number of those divisors of n of one of the forms $mx+\beta_i$.

Thus for $q=1$, $\theta=\pi/4$ we have

$$U_n=\frac{\sin n\pi/2}{\sin\pi/2}=\begin{cases}+1 & \text{if } n=4x+1\\ -1 & \text{if } n=4x+3\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$D_n=\sum_{\delta\mid n}U_\delta=E_4(n)=E_4(n\,|\,1;3).$$

In this case (6) gives us

$$E_4(n)-E_4(n-1)-E_4(n-3)+E_4(n-6)+E_4(n-10)-\cdots$$

$$=\begin{cases}(-1)^{[(k-1)/2]}[k/2] & \text{if } n=k(k-1)/2\\ 0 & \text{otherwise.}\end{cases}$$

This formula is due to Glaisher.[2]

[2] J. W. L. Glaisher, *On the function which denotes the difference between the number of $4m+1$-divisors and the number of $4m+3$-divisors of a number*, Proc. London Math. Soc. (1) vol. 15 (1884) pp. 104–122 (especially pp. 107, 110).

If $\theta=\pi/3$ we find that $U_n=0$, 1 or -1 according as n is congruent to 0, 1, or 2 modulo 3. Hence

$$D_n = E_3(n) = E_3(n \mid 1; 2).$$

In this case (6) gives

$$E_3(n) - E_3(n-3) - E_3(n-6) + E_3(n-15) + E_3(n-21) - \cdots$$
$$= \begin{cases} (2k-1)/3 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 2 \pmod 3 \\ -[(k+1)/3] & \text{if } n = k(k-1)/2 \text{ and } k \not\equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here the numbers 0, 3, 6, 15, 21, 36, 45, $\cdots$ are those triangular numbers which are multiples of 3. This formula, for the case $n\equiv 1 \pmod 3$, was given by Glaisher.[3] If $\theta=\pi/6$ then

$$D_n = E_6(n) = E_6(n \mid 1, 2; 4, 5).$$

The recurrence (6) gives, in this case,

$$E_6(n) - 2E_6(n-1) + E_6(n-3) + E_6(n-6) - 2E_6(n-10) + E_6(n-15) + E_6(n-21) - \cdots$$
$$= \begin{cases} -k & \text{if } n = k(k-1)/2 \text{ and } k \equiv 0 \pmod 3 \\ k-1 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 1 \pmod 3 \\ 1 & \text{if } n = k(k-1)/2 \text{ and } k \equiv 2 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Here the coefficient of $E_6(n-\Delta)$ is $+1$ or -2 according as Δ is or is not a multiple of 3.

In case θ is taken as $\pi/8$, the function D_n is no longer an integer, for all n, but has the value

$$D_n = E_8(n \mid 1, 3; 5, 7) + 2^{1/2}E_8(n \mid 2; 6).$$

The equation (6) now yields two formulas obtained by equating separately the rational and irrational parts of both sides. Combining these two results for simplicity we obtain the following, where

$$E_8(n) = E_8(n \mid 1, 3; 5, 7).$$

If n is even

[3] J. W. L. Glaisher, *On the function wich denotes the excess of the number of divisors of a number which* $\equiv 1$, mod 3, *over the number which* $\equiv 2$, mod 3, Proc. London Math. Soc. (1) vol. 21 (1890) pp. 395–402.

$$E_8(n) - E_8(n-1) + E_8(n-3) - E_8(n-6) - E_8(n-10) + \cdots = \begin{cases} 4[k/4](-1)^{k+1+[k/4]} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

If n is odd

$$E_8(n) + E_8(n-1) - E_8(n-3) - E_8(n-6) - E_8(n-10) - \cdots = \begin{cases} (-1)^{[k/4]}\{6[(k+1)/4] - [k/2] + (-1)^k 2\} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

In the first of these the periodic sign pattern is

$$+ - + - - + - + + - + - - + - + \cdots$$

while in the second formula it is

$$+ + - - - - + + + + - - - - + + \cdots .$$

As a by-product we have for n odd:

$$E_8'(n-1) - E_8'(n-3) - E_8'(n-15) + E_8'(n-21) + E_8'(n-45) - E_8'(n-55) - \cdots = \begin{cases} (-1)^{[k/4]}2[(k+1)/4] & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Here we have

$$E_8'(n) = E_8(n \mid 1, 2, 3; 5, 6, 7).$$

The numbers 1, 3, 15, 21, 45, $\cdots$ are the odd triangular numbers and the sign pattern is

$$+ - - + + - - + + - - + + - - + \cdots .$$

The values 0 and $\pi/2$ of θ correspond to degenerate values of Lucas U_n which becomes n and $(-1)^{n+1}n$, respectively. If we take the limiting forms of (6) in these cases we get the following two formulas

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + 9\sigma(n-10) - 11\sigma(n-15) + \cdots = \begin{cases} (-1)^k k(k-1)(2k-1)/6 & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise;} \end{cases}$$

$$\lambda(n) + \lambda(n-1) + \lambda(n-3) + \lambda(n-6) + \lambda(n-10) + \cdots = \begin{cases} n & \text{if } n \text{ is triangular} \\ 0 & \text{otherwise.} \end{cases}$$

In the first formula $\sigma(n)$ denotes the sum of the divisors of n and in

the second $\lambda(n)$ denotes the excess of the sum of the odd divisors of n over the sum of its even divisors. The first relation, apparently, was first published by Sardi[4] and later given by Halphen[5] and Glaisher.[6,7] The second is due to Glaisher[1] and Lipschitz.[8]

Two final examples of (6) in which θ is complex will be given. The first of these corresponds to $q=1$ and $\theta=\arcsin i$. Let (x_m, y_m) be the mth multiple solution of the Pell equation

$$x^2 - 2y^2 = 1,$$

and (t_m, u_m) be the mth solution of the modified Pell equation

$$t^2 - 2u^2 = -1,$$

so that

$$y_1 = 2,\ y_2 = 12,\ y_3 = 70, \cdots,\ y_{i+1} = 6y_i - y_{i-1},$$
$$t_1 = 1,\ t_2 = 7,\ t_3 = 41, \cdots,\ t_{i+1} = 6t_i - t_{i-1},$$
$$u_1 = 1,\ u_2 = 5,\ u_3 = 29, \cdots,\ u_{i+1} = 6u_i - u_{i-1}.$$

Finally let

$$S(n) = \sum_{\delta|n} y_\delta.$$

Then (6) becomes

$$t_1S(n) - t_2S(n-1) + t_3S(n-3) - t_4S(n-6) + t_5S(n-10) - \cdots = \begin{cases} (-1)^k(2ku_k - y_k)/4 & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

The case of $\theta=\arcsin(i/2)$ has to do with the Fibonacci series:

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \cdots, F_{i+1} = F_i + F_{i-1}.$$

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[4] C. Sardi, *Sulle somme dei divisori dei numeri*, Giornale di Mathematica vol. 7 (1869) pp. 112–115.

[5] G. Halphen, *Sur une formule récurrente concernant les sommes des diviseurs des nombres entiers*, Bull. Soc. Math. France vol. 5 (1877) pp. 158–160.

[6] J. W. L. Glaisher, *Theorems relating to the sum of the uneven divisors of a number*, Quart. J. Math. vol. 19 (1883) pp. 216–223 (especially p. 220).

[7] J. W. L. Glaisher, *On the sum of the divisors of a number*, Proc. Cambridge Philos. Soc. vol. 5 (1884), pp. 108–120.

[8] R. Lipschitz, *Sur les sommes des diviseurs des nombres*, C. R. Acad. Sci. Paris vol. 100 (1885) pp. 845–847. (The recurrence (7), which is misquoted in Dickson's *History of the theory of numbers* vol. 1, p. 302, was given without proof six years earlier by Glaisher, Nouvelle Correspondance Mathématique vol. 5 (1879) p. 176.)

In fact if we let

$$T(n) = \sum_{k|n} F_{2k}$$

then

$$T(n) - 4T(n-1) + 11T(n-3) - 29T(n-6) + \cdots = \begin{cases} (-1)^k k F_{2k-1} - F_{2k} & \text{if } n = k(k-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

Here the mth term of the sequence

$$1, 4, 11, 29, 76, 199, \cdots$$

is

$$F_{2m} + F_{2m-2}.$$

UNIVERSITY OF CALIFORNIA

ON PARTICULAR SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

K. L. NIELSEN AND B. P. RAMSAY

1. **Introduction.** The boundary value and characteristic value problems are classical questions in the theory of partial differential equations of elliptic type. A method for actual solution of these problems consisting of approximations by expressions $W_n = \sum_{\nu=1}^{n} \alpha_\nu^{(n)} \phi_\nu(x, y)$, where $\phi_\nu(x, y)$ are *particular solutions of the considered differential equation*, has been given by Bergman (see [1]).[1] Here the $\alpha_\nu^{(n)}$ are constants which are to be determined by the requirement that the values of W_n on the boundary approximate the given data (for details see [1]).[2]

In applying this method it is important for practical purposes to obtain a simple procedure for the construction of the particular solu-

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[1] The numbers in the brackets refer to the bibliography.

[2] This method is in a certain sense the reverse of the Rayleigh-Ritz method in which the approximating expressions satisfy the boundary conditions but do not satisfy the given equation.

tions. In this connection Bergman [2] has proved that to every equation

$$L(U) = U_{z\bar{z}} + aU_z + bU_{\bar{z}} + cU = 0, \tag{1}$$

where a, b, c are functions of $z=x+iy$ and $\bar{z}=x-iy$ and $U_z=\partial U/\partial z$, $U_{\bar{z}}=\partial U/\partial \bar{z}$, $U_{z\bar{z}}=\partial^2 U/\partial z\partial \bar{z}$, there exist functions $E(z, \bar{z}, t)$ such that

$$P(f) = \int_{-1}^{+1} E(z, \bar{z}, t) f(z(1-t^2)/2)dt/(1-t^2)^{1/2}, \tag{2}$$

where f is an arbitrary analytic function of one complex variable, will be a particular solution of $L(U)=0$.[1]

To expedite the numerical computation for practical problems we further desire that E has a simple structure so that we may easily evaluate the integral (2) for arbitrary values of z, $\bar{z}$. In this note we shall determine certain types of equations, $L(U)=0$, for which E has the simple form

$$E(z, \bar{z}, t) = \exp(Nt^n + Mt^m), \tag{3}$$

where $N=N(z, \bar{z})$ and $M=M(z, \bar{z})$. We shall further show how M and N may be determined from the coefficients a, b, c.

We note further that our results can also be successfully used for the study of the singularities of the functions U satisfying (1). In our case it is possible to show that the functions U possess certain singularities which may be characterized independently of the representation (2) by the property that these U satisfy certain ordinary differential equations in addition to $L(U)=0$.

---

[1] The equation (1) is equivalent to the system of two equations $\Delta U^{(1)}/4+AU_x^{(1)}/2+BU_y^{(1)}/2+CU_x^{(2)}/2+DU_y^{(2)}/2+c_1U^{(1)}-c_2U^{(2)}=0$, $\Delta U^{(2)}/4-CU_x^{(1)}/2-DU_y^{(1)}/2+AU_x^{(2)}/2+BU_y^{(2)}/2+c_2U^{(1)}+c_1U^{(2)}=0$ where $U=U^{(1)}+iU^{(2)}$, $c=c_1+ic_2$, $A=[(a+\bar{a})+(b+\bar{b})]/2$, $B=[(a-\bar{a})-(b-\bar{b})]/2i$, $C=-[(a-\bar{a})+(b-\bar{b})]/2i$, $D=[(a+\bar{a})-(b+\bar{b})]/2$. If $a=\bar{b}$ and c is real then these equations are completely independent of each other and it is only necessary to take the real part of (2) (see [1, p. 540]).

If $a=b=c=0$; that is, if $L(U)=\Delta U=0$, then we may take $E=1$. That is to say that every analytic function of one complex variable is a complex harmonic function. Thus analogous to the case of complex harmonic functions the totality of solutions of the equation $L(U)=0$ is the sum of two subclasses obtained by the operator P; one from analytic functions and the other from anti-analytic functions of one complex variable. Using these results and the theorem of Runge, Bergman has further shown that the representation (2) yields a method for constructing a denumerable set, S, of particular solutions (which is independent of the domain) which is complete for the totality of solutions in every star-domain with the center at the origin (see [1]).

2. **The equation $L(U)=0$ for which (2) with E of the form (3) is a class of particular solutions.** As is easily seen the equation (1) can always be transformed into the equation

$$L'(V) = V_{z\bar{z}} + BV_z + CV = 0 \tag{4}$$

(see also [2, p. 1172]). Thus it suffices to consider this equation. To simplify the notation in the following theorem we shall use $p^{(\nu)}$ to denote an arbitrary function of z *only* and $q^{(\nu)}$ to denote an arbitrary function of $\bar{z}$ *only*; we shall thus omit the arguments. k_ν shall denote constants and $p_z^{(\nu)}=dp^{(\nu)}/dz$, $q_{\bar{z}}^{(\nu)}=dq^{(\nu)}/d\bar{z}$. f shall denote an arbitrary analytic function of one complex variable.

THEOREM I. *In the equation $L'(V)=0$ let the coefficients B and C have the following forms*

$$\begin{aligned}
&\mathrm{I}_1\colon && B = 0 && \text{and} && C = -q^{(1)}q_{\bar{z}}^{(1)}/2,\\
&\mathrm{I}_2\colon && B = -p_z^{(1)} && \text{and} && C = -q^{(2)}q_{\bar{z}}^{(2)}/2,\\
&\mathrm{II}\colon && B = 0 && \text{and} && C = -3k_3zq_{\bar{z}}^{(3)}/4 - q^{(3)}q_{\bar{z}}^{(3)}/2,\\
&\mathrm{III}\colon && B = -k_1 && \text{and} && C = -q_{\bar{z}}^{(4)}q^{(4)}/2,\\
&\mathrm{IV}_1\colon && B = -q^{(5)} && \text{and} && C = (dB/d\bar{z})/2,\\
&\mathrm{IV}_2\colon && B = -q^{(6)} - p_z^{(2)} && \text{and} && C = -q_{\bar{z}}^{(6)}/2,\\
&\mathrm{V}\colon && B = 2k_2z + q^{(7)} && \text{and} && C = q_{\bar{z}}^{(7)}/2.
\end{aligned} \tag{5}$$

Then every function

$$V = \int_{-1}^{+1} \exp\,(Nt^n + Mt^m) f(z(1-t^2)/2)\,dt/(1-t^2)^{1/2} \tag{6}$$

is a particular solution of (4) *and moreover, in the respective cases*

(7)

	N	M	n	m
I_1	$z^{1/2}q^{(1)}$	0	1	0
I_2	$z^{1/2}q^{(2)}$	$p^{(1)}$	1	0
II	$k_3z^{3/2}$	$-3k_3z^{3/2}/2 - z^{1/2}q^{(3)}$	3	1
III	k_1z	$z^{1/2}q^{(4)}$	2	1
IV_1	$zq^{(5)}$	0	2	0
IV_2	$zq^{(6)}$	$p^{(2)}$	2	0
V	k_2z^2	$-2k_2z^2 - zq^{(7)}$	4	2

On the other hand if (6) *with* n, m, *and* M *given by* (7) *are solutions of* $L'(V)=0$ *for an arbitrary* f *then the coefficients* B *and* C *can be represented by the forms* (5).[4]

PROOF. The proof of this theorem is obtained by considering special cases of the general form of E, (3). A given form of E will determine a solution, V, of the partial differential equation, $L'(V)=0$, provided the coefficients B and C, satisfy the equation

$$(8)\qquad G(E) = (1-t^2)(E_{zz}) - E_z/t + 2zt(E_{zz} + BE_z + CE) = 0$$

(see [2, p. 1171]).

With the given form (3) we then desire to determine N, M, B and C in such a way that this equation is satisfied. The required derivatives are

$$(9)\qquad \begin{cases} E_z = e^{Nt^n+Mt^m}[t^nN_z + t^mM_z], \\ E_t = e^{Nt^n+Mt^m}[t^{n-1}nN + t^{m-1}mM], \\ E_z = e^{Nt^n+Mt^m}[t^mM_z + t^nN_z], \\ E_{tz} = e^{Nt^n+Mt^m}[t^{n-1}nN_z + t^{2n-1}nNN_z + t^{n-1}mM_z + t^{2m-1}mMM_z \\ \qquad + t^{n+m-1}(nMN_z + mNM_z)], \\ E_{zz} = e^{Nt^n+Mt^m}[t^nN_{zz} + t^{2n}N_zN_z + t^mM_{zz} + t^{2m}M_zM_z \\ \qquad + t^{n+m}(N_zM_z + N_zM_z)]. \end{cases}$$

These values are to be substituted into (8). When the substitution is made and the equation is divided by the common factor $\exp(Nt^n+Mt^m)$ we have an equation which is zero for an arbitrary t; thus the coefficients of each power of t must vanish. This gives us the following system of equations

$$(10)\qquad \begin{cases} t^{m-1}: & (m-1)M_z = 0 \\ t^{2m-1}: & mMM_z = 0, \\ t^{n-1}: & (n-1)N_z = 0, \\ t^{n+m-1}: & mMN_z + nNM_z = 0, \\ t^{2n-1}: & nNN_z = 0, \\ t^{m+1}: & (2Bz - m)M_z + 2zM_{zz} = 0, \\ t^{2m+1}: & 2zM_zM_z - mMM_z = 0, \\ t^{n+1}: & (2Bz - n)N_z + 2zN_{zz} = 0, \\ t^{n+m+1}: & -mMN_z - nNM_z + 2zM_zN_z + 2zM_zN_z = 0, \\ t^{2n+1}: & -nNN_z + 2zN_zN_z = 0, \\ t: & 2zC = 0. \end{cases}$$

[4] This theorem includes the special cases discussed by Bergman in [2].

For various values of m and n this system will take different forms. For many values trivial[5] solutions arise but for others non-trivial solutions exist. We thus obtain the various cases of the theorem.

We shall develop the case for $m=2$ and $n=4$. The system (10) then is

$$
(11)\quad \begin{cases} t^6: & N_z(2zN_z - 4N) = 0, \\ t^5: & M_z(2zN_z - 4N) + 4NN_z - 2MN_z + 2zN_zM_z = 0, \\ t^4: & 2MN_z + 4NM_z - 2MM_z - 4N_z + 2zM_zM_z \\ & \qquad + 2zN_{zz} + 2BzN_z = 0, \\ t^3: & 2MM_z + 3N_z - 2M_z + 2zM_{zz} + 2BzM_z = 0, \\ t: & M_z + 2Cz = 0. \end{cases}
$$

From the first equation of (11) it appears that we must consider two cases: $N_z=0$ and $N_z\neq 0$. Let us consider the first case; then this equation is satisfied. The second equation becomes

$$
(12)\qquad M_z(2zN_z - 4N) = 0.
$$

We need not consider $M_z=0$ as with $N_z=0$ this would reduce to a trivial case; we thus take $M_z\neq 0$ and from (12) we get

$$
2zN_z - 4N = 0.
$$

Solving this equation we obtain

$$
(13)\qquad N = kz^2.
$$

The third equation of (11) then reads

$$
4NM_z - 2MM_z + 2zM_zM_z = 0
$$

and since $M_z\neq 0$ and by (13) it becomes

$$
2kz^2 - M + zM_z = 0.
$$

The solution of this equation is

$$
(14)\qquad M = -(2kz^2 + zq).
$$

We now consider the fourth equation of (11) which reads

$$
(15)\qquad 2MM_z - 2M_z + 2zM_{zz} + 2BzM_z = 0.
$$

From (14) we get $M_z = -zq_z$ and $M_{zz} = -q_z$; substituting these into the equation (15) we can solve for B, getting

$$
(16)\qquad B = 2kz + q.
$$

[5] By a trivial solution we mean a solution which gives $B=C=M=N=0$.

The last equation of (11) reads

$$M_1 + 2C_x = 0$$

so that we get

$$C = q_1/2. \tag{17}$$

We thus have case V of the theorem. The other cases can be proved in a similar manner.[6] It is easily seen that all the cases satisfy equation (8).

It remains to prove that if (2) is a solution of (4); that is, $L(P(f)) = 0$, for every f then $G(E) = 0$. Let $f = (z(1-t^2)/2)^\mu$, then by taking the derivatives of $V = P(f)$ and substituting into (4) we see that it reduces to prove that from

$$\int_{-1}^{+1} \frac{(1-t^2)^\mu G(E)}{(1-t^2)^{1/2}}\, dt = 0, \qquad \mu = 0, 1, 2, \cdots, \tag{18}$$

it follows that $G(E) = 0$. This follows directly from the fact that in the integral $G(E)$ is an even function of t and the completeness of the system t^μ.

3. **A discussion of the form of** E, (3). The question of whether or not Theorem I gives all the partial differential equations (4) which have solutions (2) with E in the form (3) has not been completely answered. However, the number of non-trivial solutions for B and C is definitely limited and it is possible to give certain relations between m and n in this case by studying the relations between the exponents of t in E. For certain values of m and n, $C = 0$ and for others only trivial solutions can result.

We proceed to analyze the system (10) by comparing the exponents of t. The last equation of this system shows that in general C is zero unless m and n have values which will make at least one of the other exponents equal to 1. In order to avoid duplication of possible cases we may, without any loss of generality, make the following assumptions on the values of m and n:

(19) we shall take $n < m$ if n and m have different signs and $|n| > |m|$ in all other cases.

We thus formulate the conditions for m and n for which $C = 0$ and in Theorem III, conditions for which B and C do not vanish simultaneously.

THEOREM II. (a) *When* $n > 0$ *and* $m > 0$, $C = 0$ *unless* $m = 1$ *or* $m = 2$.

[6] A typewritten copy of the enlarged proof with a discussion of all the cases is available at the Brown University Library.

(b) *When* $n<0$ *and* $m<0$, $C=0$. (c) *When* $n<0$ *and* $m>0$, $C=0$ *unless* $m=1$, $m=2$, *or* $n=-m$.

The proof of this theorem follows directly by comparing the exponents of t in (10).

THEOREM III. *The non-trivial solutions for B and C are possible*
(i) *for all m only if* $n=2$, $n=m+2$, $n=m-2$, $n=-m$, $n=2-m$, $n=m+1$, $n=1$, $n=m/2$, $n=(m+2)/2$;
(ii) *for all n only if* $m=0$ *or* $m=2$.

PROOF. Consider the coefficient of t^{2n-1}. If no other exponent is equal to $2n-1$ then we have $nNN_1=0$. By (19) we have $n\neq 0$ and thus for this equation to be satisfied $N_1=0$. Consider the coefficient of t^{n+m-1}. If no other exponent is equal to $n+m-1$ then using the fact that $N_1=0$ this coefficient is $nNM_1=0$ and since $n\neq 0$ we must have $M_1=0$ for this equation to be satisfied. But with $N_1=0$ and $M_1=0$ we see by substitution into (10) that $B=C=0$. Hence the non-trivial solutions for B and C for general values of m and n are possible only if either of the exponents $n+m-1$ or $2n-1$ is equal to some other exponent. Checking all these possibilities and using (19) we obtain the conditions of the theorem.

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LOUISIANA STATE UNIVERSITY AND
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RESIDUE THEOREMS OF HARMONIC FUNCTIONS OF THREE VARIABLES

STEFAN BERGMAN

1. **Introduction.** The connection between analytic functions of one complex variable and harmonic functions of two real variables is used in the study of the latter functions. The analogy suggests generalizations of these methods of attack to the case of three variables.

Suppose that O is some operation which transforms the class of analytic functions of the complex variable u into the class of harmonic functions of three variables x_1, x_2, x_3. If O possesses the property that to every harmonic polynomial of the nth degree there corresponds an expression of the form $A\,[u(x_1, x_2, x_3)]^n$, where u is a linear expression in x_1, x_2, x_3, then such an operation yields a representation of spherical harmonics.

The existing representations of the spherical harmonics lead us to several operations O. The operation P described below (a generalization of the known Whittaker's formula) seems to be the most suitable for the questions considered in this note.

Let F be the class of analytic functions of complex variables u and ζ, where $u = x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta$. By the operation

$$(1.1)\quad \mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T}) = (1/2\pi i)\int_{\mathfrak{L}^1} f(u, \zeta)d\zeta, \qquad \mathfrak{X} = (x_1, x_2, x_3) \in \mathfrak{U}^3(\mathfrak{T})$$

F is transformed into the class of (complex) harmonic[1] functions (see [1, 2][2]). $\mathfrak{U}^3(\mathfrak{T})$ denotes a sufficiently small neighborhood of the point $\mathfrak{T} = (t_1, t_2, t_3)$, and $\mathfrak{L}^1$ is a closed curve in the ζ-plane.

$\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T})$ is a functional. If we carry out the integration for $\mathfrak{X} \in \mathfrak{U}^3(\mathfrak{T})$ and we continue analytically the obtained harmonic function $H(\mathfrak{X})$ to a point $\mathfrak{Y} \in \mathfrak{U}^3(\mathfrak{B})$, then the continued function is not necessarily equal to $\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{B})$. This is the reason why it is necessary to indicate the point $\mathfrak{T}$ in the neighborhood of which the integration is carried out.

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[1] Here and hereafter harmonic functions mean harmonic functions of three *real* variables x_1, x_2, x_3, except when we specifically state that we mean harmonic functions of two variables. By taking the real or imaginary part in the formulas for complex harmonic functions we obtain analogous results for real harmonic functions.

[2] The numbers in brackets refer to the bibliography.

E [· · ·] means the set of points satisfying the relations indicated in the brackets. Superscripts over a manifold indicate its dimensionality.

The analogy suggests further consideration of harmonic vectors $\mathfrak{H}(\mathfrak{X})=H^{(1)}(\mathfrak{X})\mathfrak{i}_1+H^{(2)}(\mathfrak{X})\mathfrak{i}_2+H^{(3)}(\mathfrak{X})\mathfrak{i}_3$, in addition to a single function, where $H^{(k)}$ are harmonic functions connected by certain differential relations.[3] $\mathfrak{i}_k$ is the unit vector in the x_k direction.

In §§2 and 3 we shall consider harmonic vectors

$$\mathfrak{H}(\mathfrak{X}) = \mathfrak{P}(f, \mathfrak{Q}^1, \mathfrak{T}) \equiv \mathrm{P}[f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_1 + \mathrm{P}[(i/2)(\zeta + \zeta^{-1})f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_2 + \mathrm{P}[(1/2)(\zeta - \zeta^{-1})f, \mathfrak{Q}^1, \mathfrak{T}]\mathfrak{i}_3. \tag{1.2}$$

Clearly, curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$ at every point at which all components of $\mathfrak{H}$ are regular.

Consider vectors $\mathfrak{H}(\mathfrak{Y})=H^{(1)}(\mathfrak{Y})\mathfrak{i}_1+H^{(2)}(\mathfrak{Y})\mathfrak{i}_2$, $\mathfrak{Y}=(y_1, y_2)$ of two real variables y_1, y_2 for which we have curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$. $\int_{\mathfrak{Y}^1}\mathfrak{H}\cdot d\mathfrak{Y}=0$ if $\mathfrak{Y}^1$ is a closed curve, and if $\mathfrak{Y}^1$ can be shrunk to a point in a regularity domain of $\mathfrak{H}$. ($\cdot$ means the scalar product.) If singularities of $\mathfrak{H}$ lie inside of $\mathfrak{Y}^1$ or $\mathfrak{H}$ is a many valued function the residue theorem yields well known relations.

Let $\mathfrak{R}^2$ be some simply connected domain in which $\mathfrak{Y}^1$ lies, and let $\mathfrak{H}=\mathfrak{H}_1+\sum_{k=1}^{n}\mathfrak{S}_k$, where $\mathfrak{H}_1$ is a vector which is regular in $\mathfrak{R}^2$. Then

$$\int_{\mathfrak{Y}^1}\mathfrak{H}\cdot d\mathfrak{Y} = \int_{\mathfrak{Y}^1}\mathfrak{H}_1\cdot d\mathfrak{Y} + \sum_{k=1}^{n}\int_{\mathfrak{Y}^1}\mathfrak{S}_k\cdot d\mathfrak{Y}.$$

In this connection we make the trivial simplification that we limit ourselves to the case of suitable standard vectors $\mathfrak{S}$ which possess one singular point (or one connected singular line in the case of three variables).

If $\mathfrak{S}(\mathfrak{Y})=\mathrm{Im}\ [(z-\alpha)^{-1}]\mathfrak{i}_1+\mathrm{Re}\ [(z-\alpha)^{-1}]\mathfrak{i}_2$ then $\int_{\mathfrak{Y}^1}\mathfrak{S}(\mathfrak{Y})\cdot d\mathfrak{Y}=2\pi$ if α lies inside of $\mathfrak{Y}^1$. Since $\mathfrak{S}$ can be considered as a harmonic vector in three variables the above relation may be formulated in the following form. Let $\mathfrak{S}(\mathfrak{X})$ be a harmonic vector which in appropriate coordinates y_1, y_2, y_3 may be written

$$\mathfrak{S} = \mathrm{Im}\,(y_1 + iy_2 - \alpha)^{-1}\mathfrak{i}_1 + \mathrm{Re}\,(y_1 + iy_2 - \alpha)^{-1}\mathfrak{i}_2 + 0\cdot\mathfrak{i}_3.$$

Then $\int_{\mathfrak{Y}^1}\mathfrak{S}(\mathfrak{X})\cdot d\mathfrak{X}=2\pi$ if $\mathfrak{Y}^1$ cannot be reduced to a point without cutting E $[y_3=0, y_1+iy_2=\alpha]$. As it is known (see [1, 2]) there exist algebraic harmonic functions with singularity lines or points [see

[3] Harmonic vectors are met in physical applications especially in the theory of a perfect incompressible fluid.

The above mentioned differential relations can vary in nature depending upon the question which has to be considered. We limit ourselves in this paper to special relations, namely, curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$ in §§2 and 3 and (4.3) in §4. However, in many instances with certain changes we may extend our results to much more general relations.

(2.5) p. 166] which cannot be represented as harmonic functions of two variables. In §§2 and 3 we introduce harmonic vectors $\mathfrak{S}(\mathfrak{x})$ with such singularities and consider

$$\sum_k \int_{\mathfrak{x}_k^{(1)}}^{\mathfrak{x}_k^{(2)}} \mathfrak{S}_k(\mathfrak{x}) \cdot d\mathfrak{x}.$$

Here $\mathfrak{S}_k$ are branches of $\mathfrak{S}$ and $\mathfrak{x}_k^{(\mathrm{H})}$ points which satisfy certain algebraic relations. We show that the above expression is equal to a quantity connected with the singularity, which may be interpreted as the residue of $\mathfrak{S}$.[4]

A theorem of related character concerning surface integrals is given in §4.

As pointed out before, the operation P transforms a polynomial in u and ζ into a harmonic polynomial of the same degree.

As we shall see below a rational function f is in general transformed into an algebraic harmonic function of x_1, x_2, x_3. The question arises whether every algebraic harmonic function $A(x_1, x_2, x_3)$ can be represented in the form $\mathrm{P}(f, \mathfrak{Q}^1, \mathfrak{T})$ where f is a rational function in u and ζ. This question has not been answered. However, from the result of [2, p. 637] follows that $A(x_1, x_2, x_3)$ can be represented in the above form with $f = \int_0^1 a(u, \zeta, T)dT$, a being an algebraic function in u, ζ, T.

2. **Singularities of harmonic functions.** The introduction of the operator (1.1) gives a simple procedure for the construction of standard functions with singularities. Suppose that $f(u, \zeta)$ has poles, say for instance $f = p/q$, where p and q are polynomials in u and ζ. For simplicity's sake we shall suppose that $q = [u - \phi(\zeta)]\zeta^s$ where $\phi(\zeta)$ is a polynomial in ζ and s a non-negative integer.

In order to study the functional

$$(2.1)\quad \mathrm{P}[p/q, \mathfrak{Q}^1, \mathfrak{T}] = \frac{1}{2\pi i}\int_{\mathfrak{Q}^1} \frac{p(u, \zeta)}{q(u, \zeta)}\, d\zeta, \quad \mathfrak{x} \in \mathfrak{U}^3(\mathfrak{T}), \mathfrak{x} = (x_1, x_2, x_3),$$

as a function of $\mathfrak{T}$ it is useful to consider (2.1) in the (five-dimensional) space x_1, x_2, x_3, ξ, η; where $\zeta = \xi + i\eta$.

$$(2.2)\quad \mathfrak{Z}^3 = \mathrm{E}\{q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0\}$$

is the singularity manifold of p/q. $\mathfrak{Z}^3$ can also be written in the form

$$(2.3)\qquad \mathfrak{Z}^3 = \mathrm{E}[\zeta = \zeta_\nu(\mathfrak{x}), \nu = 1, 2, \cdots, n]$$

where ζ_ν are algebraic functions of x_1, x_2, x_3 and n the degree of q in ζ.

[4] This result has a certain similarity with Abel's theorem in the theory of integrals of algebraic functions.

At every point $\mathfrak{X}$, except those which lie in

$$\mathfrak{U}^1 = \mathrm{E}\left\{\prod_{k\neq s}[\zeta_s(\mathfrak{X}) - \zeta_k(\mathfrak{X})]^2 = 0\right\}$$

we have n distinct branches $\mathfrak{Z}_\nu^2 = \mathrm{E}\ [\zeta = \zeta_\nu(\mathfrak{X})], \nu = 1, 2, \cdots, n$ of $\mathfrak{Z}^2$.

Suppose $\mathfrak{D} \in \mathfrak{U}^1$, and $\mathfrak{C}_\nu^1$, $\nu = 1, 2, \cdots, n$, are chosen in such a way that one and only one $\zeta_\nu(\mathfrak{D})$ lies inside of each $\mathfrak{C}_\nu^1$. Then by the residue theorem $\mathrm{P}[p/q, \mathfrak{C}_\nu^1, \mathfrak{D}] = S_\nu(\mathfrak{X})$, where S_ν is the corresponding branch of

$$S(\mathfrak{X}) = \frac{p[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]}{\partial\{q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]\}/\partial\zeta} \tag{2.4}$$

$$q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0.$$

$S(\mathfrak{X})$ is an algebraic function which possesses n branches. We denote the (multiply covered) x_1, x_2, x_3-space in which $S(\mathfrak{X})$ is univalent by $\mathfrak{R}^3$; its sheets (which are supposed to be obtained by appropriate cuts), by[5] $\mathfrak{R}_\nu^3$. The singularity line of $S(\mathfrak{X})$ is

$$q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0,$$

$$\frac{\partial q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta]}{\partial\zeta} = 0. \tag{2.5}$$

EXAMPLE. $\mathrm{P}[(u - ai)^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}]$, $\mathfrak{Q}^1 = \mathrm{E}[\zeta = e^{it}, 0 \leqq t \leqq 2\pi]$, $a \geqq 0$, $t_1 \geqq 0$ is a two valued function. (See [1, p. 655].) If $a > 0$ then the singularity line $\mathrm{E}[x_1 = 0, x_2^2 + x_3^2 = a^2]$ is the branch line. For $a = 0$ this branch line shrinks to a point, both branches $S_k(\mathfrak{X}) = (-1)^{k+1}/r$ for $k = 1, 2$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, are connected only at the origin.

We further note that $\mathrm{P}[(1/2)(\zeta + \zeta^{-1})u^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}] = -ir^{-1}\cos\phi \cdot \tan(\theta/2)$, $\mathrm{P}[(1/2i)(\zeta - \zeta^{-1})u^{-1}\zeta^{-1}, \mathfrak{Q}^1, \mathfrak{T}] = -ir^{-1}\sin\phi\tan(\theta/2)$ for $t_1 > 0$. Here r, θ, ϕ are polar coordinates.

In study of the representation (2.1) the points $\mathfrak{X}$ which satisfy the equation $q[(x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta), \zeta] = 0$ (for an appropriate ζ) are of importance. In this connection it is useful to introduce certain curves, $\mathfrak{L}^1(\zeta)$.

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[5] We suppose here that $\zeta = 0$ lies outside of $\mathfrak{C}_\nu^1$. Otherwise, and also for the evaluation of (2.1) if the degree of q (in u) is higher than 1, see [2, pp. 644–650], especially the formulas (7) and (10).

We note that the $\mathfrak{R}_\nu^3$ are not necessarily connected with each other in the *real* x_1, x_2, x_3-space.

By the correspondence

$$(2.6)\qquad \mathfrak{X} \to u = x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta$$

we associate to every point $\mathfrak{X}$ of the x_1, x_2, x_3-space a point u of the complex plane. On the other hand to the point $A+iB$ of the u-plane there corresponds a line $\mathcal{P}^1(\zeta, A+iB)$ in the x_1, x_2, x_3-space. If we write $x_2+ix_3=\rho e^{i\varphi}$, $\zeta=\tau e^{i\alpha}$ then from $u=A+iB$ it follows that $ie^{i(\varphi-\alpha)}\tau^{-1}+2x_1\rho^{-1}+ie^{-i(\varphi-\alpha)}\tau=2\rho^{-1}(A+iB)$ or

$$(2.7)\qquad \mathcal{P}^1(\tau e^{i\alpha}, A+iB) = \mathrm{E}\left[\rho^2 = 4\left(\frac{B}{\tau+\tau^{-1}}\right)^2 + 4\left(\frac{A-x_1}{\tau-\tau^{-1}}\right)^2,\ \varphi = \alpha + \arccos\frac{2B\rho^{-1}}{\tau+\tau^{-1}}\right].$$

We put

$$(2.8)\qquad \mathcal{L}^1(\zeta) = \mathcal{P}^1[\zeta, \phi(\zeta)]$$

where $\phi(\zeta)$ is the polynomial introduced on p. 165. For $\mathfrak{X}\in\mathcal{L}^1(\zeta_0)$ we have

$$q(u, \zeta_0) = q\left[(x_1 + (i/2)(x_2 + ix_3)\zeta_0^{-1} + (i/2)(x_2 - ix_3)\zeta_0, \zeta_0\right]$$
$$= \zeta_0^s[u_0 - \phi(\zeta_0)] = 0;$$

therefore

$$(2.9)\qquad \mathcal{L}^1(\zeta_0) = \mathcal{Z}^3 \cap \mathrm{E}(\zeta = \zeta_0).$$

Since in[6] $\mathcal{R}_1^3$, $\mathcal{Z}^3 = \mathfrak{S}_{r=1}^n \mathcal{Z}_r^3$, we can write $\mathcal{L}^1(\zeta) = \mathfrak{S}_{r=1}^n \mathcal{L}_r^1(\zeta)$ where $\mathcal{L}_r^1(\zeta_0) = \mathcal{Z}_r^3 \cap \mathrm{E}[\zeta = \zeta_0]$.

Let $\mathcal{Q}^1$ be a closed curve in the ζ-plane. We suppose (for simplicity's sake) that one and only one point $\zeta_k(\mathfrak{D})$, say $\zeta_1(\mathfrak{D})$, lies inside of $\mathcal{Q}^1$. Then $\mathrm{P}[p/q, \mathcal{Q}^1, \mathfrak{T}] = S_1(\mathfrak{X})$ for $\mathfrak{T}=\mathfrak{D}$. However, if $\mathfrak{T}$ varies in $\mathcal{R}_1^3$ new branches $\mathcal{Z}_r^3$ can slip in or out from $\mathcal{Q}^1$. Since $\mathcal{Z}^3$ is a continuous manifold, $\mathcal{Z}_r^3$ can enter $\mathcal{Q}^1$ only for such values of $\mathfrak{X}$ for which $\mathcal{Z}_r^3$ has intersection with $\mathrm{E}[\zeta\in\mathcal{Q}^1]$, that is for $\mathfrak{X}\in\mathcal{Z}_r^3 \cap \mathrm{E}[\zeta\in\mathcal{Q}^1] = \mathfrak{S}_{\zeta\in\mathcal{Q}^1}\mathcal{L}_r^1(\zeta) = \mathcal{M}_r^2(\mathcal{Q}^1)$. Thus, if the point $\mathfrak{T}$ in the operator (2.1) moves and passes through $\mathcal{M}^2(\mathcal{Q}^1)$, the integral $\mathrm{P}[p/q, \mathcal{Q}^1, \mathfrak{T}]$ has a jump,[7] it increases or decreases by $S_r(\mathfrak{X})$.

[6] $\mathcal{R}_1^3$ means the sheet of $\mathcal{R}^3$ in which $S_1(\mathfrak{X})$ is defined. (See p. 166). $\mathfrak{S}$ denotes the topological sum. $\cap$ means the intersection.

We suppose again that the point $\zeta=0$ lies neither inside ζ_1^1 nor inside of $\mathcal{Q}^1$. Otherwise we must take into account the residue at $\zeta=0$ since the integrand of (2.1) can have a pole at this point.

[7] The surfaces $\mathcal{M}^2(\mathcal{Q}^1)$ are denoted as surfaces of separation of the representation (2.1).

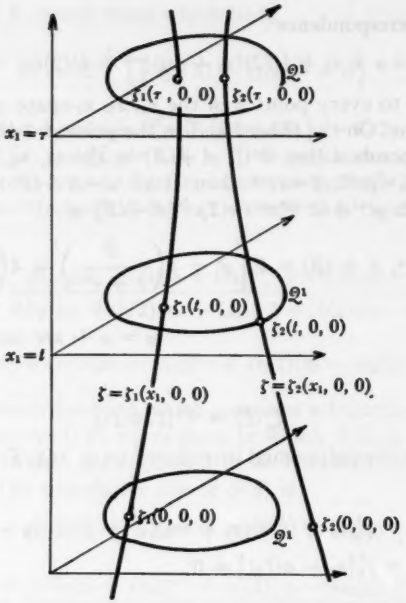

FIG. 1. The intersection of $\mathfrak{Z}^2$ with $\mathrm{E}[x_2=0, x_3=0]$

The curves $\mathfrak{L}^1(\zeta)$ play also a role in another connection. Consider a closed curve $\mathfrak{I}^1$ in the schlicht x_1, x_2, x_3-space. To $\mathfrak{I}^1$ there corresponds by (2.6) a closed curve $\mathfrak{F}^1(\zeta)$ in the u-plane. In the following we consider the case where the point $u=\phi(\zeta)$ lies inside of $\mathfrak{F}^1(\zeta)$. If we construct the surface $\mathfrak{K}^2=\mathfrak{S}_{Z\in\mathfrak{I}^1(\zeta)}\mathfrak{P}^1(\zeta, Z)$, then $\phi(\zeta)$ lies inside of $\mathfrak{F}^1(\zeta)$ if and only if $\mathfrak{K}^2$ cannot be reduced to a line without cutting $\mathfrak{L}^1(\zeta)=\mathfrak{P}^1(\zeta, \phi(\zeta))$. In general, this will happen if and only if the curve $\mathfrak{I}^1$ cannot be reduced to a point without cutting $\mathfrak{L}^1(\zeta)$.

The statement that $\mathfrak{I}^1$ cannot be reduced to a point without cutting $\mathfrak{L}^1(\zeta)$ we shall understand in the above sense.

The essential idea of the method of attack which will be used in the next section is the following:

Let $\mathfrak{I}^1$ be a closed curve in the x_1, x_2, x_3-space. If we divide it in an appropriate way, $\mathfrak{I}^1=\mathfrak{S}_\nu\mathfrak{I}^1_\nu$, then, by the operation P, the differential $f(u, \zeta)du$ will be transformed into $\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}$, for $\mathfrak{X}\in\mathfrak{I}^1_\nu$, that is, we have $[\int_{\mathfrak{Q}^1} f(u, \zeta)d\zeta]du=\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}$. $\mathfrak{I}^1_\nu$ means that part of $\mathfrak{I}^1$ within whose points $f(u, \zeta)$ will be transformed into the branch $\mathfrak{S}_\nu(\mathfrak{X})$.

Here $\mathfrak{S}_\nu(\mathfrak{X})$ is an appropriate branch of the algebraic harmonic

vector $\mathfrak{S}(\mathfrak{X})$. (See (3.1).) Thus

$$(2.10)\qquad (1/2\pi i)\int_{\mathfrak{J}^1}\left\{\int_{\mathfrak{Q}^1}[f(u,\zeta)d\zeta]\right\}du = \sum_{\nu}\int_{\mathfrak{J}^1_\nu}\mathfrak{S}_\nu(\mathfrak{X})\cdot d\mathfrak{X}.$$

Changing the order of the integration in the left side member and applying the residue theorem for functions of one complex variable, we evaluate (2.10).

3. **Residue formula for line integrals.** Let $\mathfrak{J}^1$ be a closed differentiable curve in the schlicht x_1, x_2, x_3-space. We place it in $\mathfrak{R}_1^3$ and suppose that $\mathfrak{J}^1$ intersects the cuts which bound $\mathfrak{R}_1^3$ at one (and only one) point $\mathfrak{E}$. Thus $\mathfrak{J}^1$ is an open connected curve in $\mathfrak{R}_1^3$. Its end points (which coincide in the schlicht space with $\mathfrak{E}$) are denoted by $\mathfrak{E}_1$ and $\mathfrak{E}_2$.

Let $\mathfrak{Q}^1$ be a closed curve in the ζ-plane. We suppose (for simplicity's sake) that $\zeta_1(\mathfrak{E}_1)$ lies inside and $\zeta_\nu(\mathfrak{E}_1)$, $\nu=2, \cdots, n$, lie outside of $\mathfrak{Q}^1$. (See Figure 1.)

THEOREM. *Let*

$$(3.1)\qquad \mathfrak{S}(\mathfrak{X}) = \left[\frac{p(u,\zeta)}{\partial q(u,\zeta)/\partial\zeta}\right]\left[\mathfrak{i}_1 + \frac{i(\zeta+\zeta^{-1})}{2}\mathfrak{i}_2 + \frac{(\zeta-\zeta^{-1})}{2}\mathfrak{i}_3\right],$$

$$u \equiv [x_1 + (i/2)(x_2+ix_3)\zeta^{-1} + (i/2)(x_2-ix_3)\zeta]$$

where $\zeta=\zeta(x_1, x_2, x_3)$ *is a root of the equation*

$$(3.2)\qquad q(u,\zeta) \equiv [u-\phi(\zeta)]\zeta^s = 0,$$

be a harmonic vector, the branches of which we denote by $\mathfrak{S}_\nu(\mathfrak{X})$, $\nu=1, 2, \cdots, n$. *The corresponding sheets of the multiply covered space of* $\mathfrak{S}(\mathfrak{X})$ *may be denoted by* $\mathfrak{R}_\nu^3$.

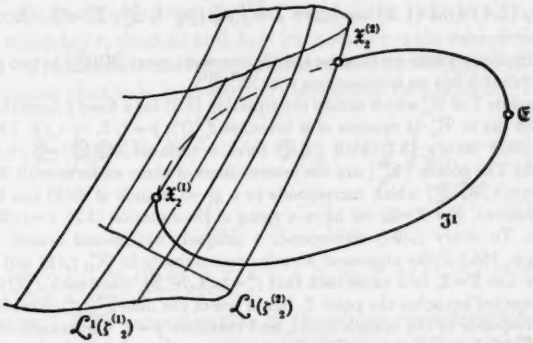

FIG. 2. The curve $\mathfrak{J}^1$ and the surface $\mathfrak{M}_1^3(\mathfrak{Q}^1)$

Let $\mathfrak{I}^1$ and $\mathfrak{Q}^1$ be curves in the x_1, x_2, x_3-space and in the ζ-plane, respectively, with the above described properties.

Let finally[8] $\{\mathfrak{X}_{\nu_\mu}^{(k)}\}$, $k=1, 2$; $\mu=1, 2, \cdots, p$ *be the points of* $\mathfrak{I}^1$ *which satisfy the equation* (3.2) *for appropriate values*[9] $\zeta=\zeta_\nu^{(k)}$, $\zeta_\nu^{(k)}\in\mathfrak{Q}^1$.

Then

$$(3.3)\qquad \int_{\mathfrak{C}_1}^{\mathfrak{C}_2}\mathfrak{S}_1(\mathfrak{X})\cdot d\mathfrak{X}+\sum_{\mu=1}^{p}\int_{\mathfrak{X}_{\nu_\mu}^{(1)}}^{\mathfrak{X}_{\nu_\mu}^{(2)}}\mathfrak{S}_{\nu_\mu}(\mathfrak{X})\cdot d\mathfrak{X}=2\pi i\int_{\mathfrak{S}\mathfrak{N}_{\mathfrak{U}}^1}p[\phi(\zeta),\zeta]d\zeta$$

where $\mathfrak{N}_{\mathfrak{U}}^1$ are the intervals of $\mathfrak{Q}^1$, for which the filaments $\mathfrak{L}^1(\zeta)$, of $\mathfrak{M}^2(\mathfrak{Q}^1)$ lie inside of $\mathfrak{I}^1$; $\zeta_{\nu_\mu}^{(k)}$ are their end points.

REMARK. We suppose here that the point $\zeta=0$ does not lie inside of $\mathfrak{Q}^1$. Otherwise the components of the vector assume a form slightly different from that indicated in (3.1). These components in this case can be easily computed using the formulas of [2, §2].

PROOF. We consider the double integral

$$I=\int_{\mathfrak{I}^1}\int_{\mathfrak{Q}^1}\frac{p(u,\zeta)d\zeta du}{q(u,\zeta)}.$$

The integrand is an absolutely integrable function; we can replace the double integral by the iterated one and change the order of the integration. Thus,

$$I=\int_{\mathfrak{I}^1}\left\{dx_1\int_{\mathfrak{Q}^1}pq^{-1}d\zeta+dx_2\int_{\mathfrak{Q}^1}(i/2)(\zeta+\zeta^{-1})pq^{-1}d\zeta\right.$$
$$\left.+\,dx_3\int_{\mathfrak{Q}^1}(1/2)(\zeta-\zeta^{-1})pq^{-1}d\zeta\right\}.$$

By (3.1), (2.4) and (1.2) we have $I=\int_{\mathfrak{I}^1}\mathfrak{P}(pq^{-1}, \mathfrak{Q}^1, \mathfrak{X})\cdot d\mathfrak{X}$. According

[8] For simplicity's sake we suppose that $\mathfrak{I}^1$ intersects every $\mathfrak{M}^2(\mathfrak{Q}^1)$ in two points (if at all), and that it has no intersections with $\mathfrak{M}_1^2(\mathfrak{Q}^1)$.

[9] The points $\mathfrak{X}$ of $\mathfrak{R}_1^3$ which satisfy the equation (3.2) for a fixed ζ form the part of $\mathfrak{L}^1(\zeta)$ which lies in $\mathfrak{R}_1^3$. It consists of n branches $\mathfrak{L}_\nu^1(\zeta)$, $\nu=1, 2, \cdots, n$. The points $\mathfrak{X}$ of $\mathfrak{R}_1^3$ which satisfy (3.2) with $\zeta\in\mathfrak{Q}^1$ form n surfaces $\mathfrak{M}_\nu^2(\mathfrak{Q}^1)=\mathfrak{S}_{\zeta\in\mathfrak{Q}^1}\mathfrak{L}_\nu^1(\zeta)$. (See p. 167.) The points $\{\mathfrak{X}_{\nu_\mu}^{(k)}\}$ are the intersections of these surfaces with $\mathfrak{I}^1$.

The branch $\mathfrak{M}_\nu^2(\mathfrak{Q}^1)$ which corresponds to a given branch of $\mathfrak{S}(\mathfrak{X})$ can be determined as follows. For $\mathfrak{X}=\mathfrak{C}_1$ we have n roots of the equation (3.2), $\zeta=\zeta_\nu(\mathfrak{C}_1)$, $\nu=1, 2, \cdots, n$. To every $\zeta_\nu(\mathfrak{C}_1)$ corresponds a uniquely determined branch $\mathfrak{S}_\nu(\mathfrak{X})$ of $\mathfrak{S}(\mathfrak{X})$. (See p. 166.) If the argument $\mathfrak{X}$ varies continuously in $\mathfrak{R}_1^3$, $\zeta_\nu(\mathfrak{X})$ will move in the ζ-plane. Let $\mathfrak{X}=\mathfrak{X}_\tau$ be a value such that $\zeta_\nu^0=\zeta_\nu(\mathfrak{X}_\tau)\in\mathfrak{Q}^1$. Since each $\mathfrak{L}^1(\zeta)$ consists of n disconnected branches the point $\mathfrak{X}_\tau$ lies in one of the lines $\mathfrak{L}_\rho^1(\zeta_\rho^0)$. This branch of $\mathfrak{L}^1(\zeta_\nu^0)$ corresponds to the branch $\mathfrak{S}_\nu(\mathfrak{X})$, and therefore $\rho=\nu$. The branch of $\mathfrak{M}^2(\mathfrak{Q}^1)$ to which $\mathfrak{L}^1(\zeta_\nu^0)$ belongs becomes $\mathfrak{M}_\nu^2(\mathfrak{Q}^1)$.

to the results on p. 167, $\mathfrak{P}(pq^{-1}, \mathfrak{Q}^1, \mathfrak{X}) = \mathfrak{P}(pq^{-1}, \mathfrak{Q}^1, \mathfrak{E}_1) + \sum_{\nu=1}^{\mu} \mathfrak{S}_\nu(\mathfrak{X})$ for all points of $\mathfrak{Z}^1$ which lie between $\mathfrak{X}_{\nu_\mu}^{(1)}$ and $\mathfrak{X}_{\nu_\mu}^{(2)}$, $\mu = 1, 2, \cdots$, p. Whence

$$I = \int_{\mathfrak{E}_1}^{\mathfrak{E}_2} \mathfrak{S}_1(\mathfrak{X}) \cdot d\mathfrak{X} + \sum_{\mu=1}^{p} \int_{\mathfrak{X}_{\nu_\mu}^{(1)}}^{\mathfrak{X}_{\nu_\mu}^{(2)}} \mathfrak{S}_{\nu_\mu}(\mathfrak{X}) \cdot d\mathfrak{X}.$$

Changing the order of integration we obtain

$$I = \int_{\mathfrak{Q}^1} d\zeta \left[\int_{\mathcal{T}^1(\zeta)} [p(\tau, \zeta)/q(\tau, \zeta)] d\tau \right] = \int_{\mathfrak{Q}^1} J(\zeta) d\zeta.$$

$\mathcal{T}^1(\zeta)$ is a closed curve in the complex τ-plane and $q = [\tau - \phi(\zeta)]\zeta^s$. (See p. 167.) $J(\zeta) = 2\pi i p[\phi(\zeta), \zeta]$ or 0 if $\tau = \phi(\zeta)$ lies inside or outside of $\mathcal{T}^1(\zeta)$, respectively.

According to the considerations on p. 168, $\phi(\zeta)$ lies inside of $\mathcal{T}^1(\zeta)$ if and only if $\mathfrak{Z}^1$ cannot be reduced to a point without cutting $\mathcal{L}^1(\zeta)$. I therefore is equal to the right-hand member of (3.3), which completes our proof.

EXAMPLE. We take $p = 1$, $q = \zeta u$, $\mathfrak{Q}^1 = \mathrm{E}[\zeta = e^{it}, 0 \leqq t \leqq 2\pi]$. The vector $\mathfrak{S}(\mathfrak{X})$ (see (3.1)) which we obtain in this case has two branches $\mathfrak{S}_1(\mathfrak{X}) = r^{-1}[\mathfrak{i}_1 + x_2(x_1 + r)^{-1}\mathfrak{i}_2 + x_3(x_1 + r)^{-1}\mathfrak{i}_3]$ and $\mathfrak{S}_2(\mathfrak{X}) = -r^{-1}[\mathfrak{i}_1 + x_2(x_1 - r)^{-1}\mathfrak{i}_2 + x_3(x_1 - r)^{-1}\mathfrak{i}_3]$, $r^2 = x_1^2 + x_2^2 + x_3^2$. The branches $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are not connected in the real x_1, x_2, x_3-space. They are of a different form from (3.1) since $\zeta = 0$ lies inside of $\mathfrak{Q}^1$. We obtain $\mathfrak{S}_1(\mathfrak{X})$ and $\mathfrak{S}_2(\mathfrak{X})$ choosing the x_1 component of $\mathfrak{T}$ to be positive or negative, respectively. Further we have in this case:

$\mathcal{L}^1(\zeta) = \mathrm{E}[x_1 = 0, x_2 \cos t + x_3 \sin t = 0]$ where $t = -i \log \zeta$; $\mathcal{M}^2(\mathfrak{Q}^1) = \mathfrak{S}\, \mathcal{L}^1(\zeta) = \mathrm{E}[x_1 = 0]$.

Let $\mathfrak{Z}^1 = \mathfrak{Z}_1^1 + \mathfrak{Z}_2^1$ be a closed curve in the x_1, x_2, x_3-space, where $\mathfrak{Z}_1^1$ denotes the part which lies in $\mathrm{E}[x_1 > 0]$, $\mathfrak{Z}_2^1$ the part in $\mathrm{E}[x_1 < 0]$. By $\mathfrak{X}_1 = (r_1 \cos \phi_1 \mathfrak{i}_2 + r_1 \sin \phi_1 \mathfrak{i}_3)$ and $\mathfrak{X}_2 = (r_2 \cos \phi_2 \mathfrak{i}_2 + r_2 \sin \phi_2 \mathfrak{i}_3)$, $0 < \phi_1 < \pi/2$, $\pi/2 < \phi_2 < \pi$, we denote the intersections of $\mathfrak{Z}^1$ with $\mathcal{M}^2(\mathfrak{Q}^1)\,(= \mathrm{E}[x_1 = 0])$. (We suppose that this intersection consists of two points.) Then

$$\sum_{\nu=1}^{2} \int_{\mathfrak{Z}_\nu^1} \mathfrak{S}_\nu(\mathfrak{X}) \cdot d\mathfrak{X} = (1/2\pi i) \int_{\mathfrak{Q}^1} \zeta^{-1} d\zeta \left[\int_{\mathcal{T}^1(\zeta)} u^{-1} du \right]$$

$$= \int_{\mathcal{N}_1^1} \zeta^{-1} d\zeta - \int_{\mathcal{N}_2^1} \zeta^{-1} d\zeta,$$

where $\mathcal{N}_1^1 = \mathrm{E}[\zeta = e^{it}, \phi_1 + \pi/2 \leqq t \leqq \phi_2 + \pi/2]$, $\mathcal{N}_2^1 = \mathrm{E}[\zeta = e^{it}, \phi_1 + 3\pi/2 \leqq t \leqq \phi_2 + 3\pi/2]$.

REMARK 1. The generalization of the above considerations if q is a polynomial of a higher degree than 1 presents no difficulties. (See [2].)

We note further that our considerations can be generalized without essential difficulties to the case where p and q are entire functions.

REMARK 2. We supposed that $\mathfrak{L}^1$ is a closed curve. However, our method can be applied also if $\mathfrak{L}^1$ is an open curve. An operator $\mathbf{P}(f, \mathfrak{L}^1, \mathfrak{T})$ with $\mathfrak{L}^1$ being an open curve transforms rational functions f into harmonic functions with logarithmic singularities. The fact that the latter harmonic functions were not studied in detail prevented us from discussing this case here.

4. **Residue formulas for surface integrals. Generalized Cauchy's formula.** There exists another possibility of generalization of the residue formula, namely, in considering surface integrals.

Various relations for surface integrals can be obtained using Green's formula. In particular a well known formula for the value of the function in the domain in terms of boundary values of the function and its derivative can be established in this way.

In the case of two variables for a harmonic vector $\mathfrak{H}$ with curl $\mathfrak{H}=0$, div $\mathfrak{H}=0$, $\mathfrak{H}(\mathfrak{X})$ may be expressed in terms of the boundary values of its components. (In Cauchy's integral formula for analytic functions which may be obtained in this way, the derivatives disappeared.)

It is clear that this procedure can be applied to harmonic vectors of several variables, provided that corresponding differential relations exist between the derivative of the components.[10] As an illustration we shall derive a formula which can be considered as a generalization of Cauchy's formula.

Let $H(\mathfrak{X})$ be a harmonic function which is regular in a (schlicht) domain $\mathfrak{D}^3$ with twice continuously differentiable boundary $\mathfrak{B}^2$. Let G_ν be twice continuously differentiable functions, $\nu=1, 2, \cdots, n$, such that

$$\nabla H = \sum_{\nu=1}^{n} \nabla G_\nu \times \mathfrak{a}_\nu, \tag{4.1}$$

($\times$ means vector product), where $\mathfrak{a}_\nu = a_{\nu 1}\mathfrak{i}_1 + a_{\nu 2}\mathfrak{i}_2 + a_{\nu 3}\mathfrak{i}_3$ are vectors with constant coefficients. Then

$$H(\mathfrak{X}) = (1/4\pi) \int\int_{\mathfrak{B}^2} \left[H\nabla r^{-1} + \nabla r^{-1} \times \sum_{\nu=1}^{n} G_\nu \mathfrak{a}_\nu \right] \cdot d\mathfrak{d} \tag{4.2}$$

where $d\mathfrak{d} = dx_2 dx_3 \mathfrak{i}_1 + dx_1 dx_3 \mathfrak{i}_2 + dx_1 dx_2 \mathfrak{i}_3$.

[10] In particular this procedure has been used in the theory of functions of several complex variables to obtain generalizations of Cauchy's residue formula. See Bergman, *Partial differential equations*, Publication of Brown University, 1941 Supplementary Note III pp. SIII 8–SIII 9.

See also the further remarks concerning the other possibilities of generalizations of Cauchy's formulas mentioned there and Bergman, Revista de Ciencias, Lima, Peru, vol. 44 (1942) pp. 377–394 especially p. 391.

PROOF. By Green's formula,

$$H(\mathfrak{X}) = (1/4\pi)\left[\int\int_{\mathfrak{B}^2} H\nabla r^{-1}\cdot d\sigma - \int\int_{\mathfrak{B}^2} r^{-1}\nabla H\cdot d\sigma\right]$$
$$= (1/4\pi)(I_1 - I_2).$$

By (4.1)

$$I_2 = \int\int_{\mathfrak{B}^2}\left[\frac{1}{r}\left(\frac{\partial H}{\partial x_1}dx_2dx_3 + \frac{\partial H}{\partial x_2}dx_1dx_3 + \frac{\partial H}{\partial x_3}dx_1dx_2\right)\right]$$
$$= \int\int_{\mathfrak{B}^2} r^{-1}\Big\{\sum_{\nu=1}^{n}\left(a_{3\nu}\frac{\partial G_\nu}{\partial x_2} - a_{2\nu}\frac{\partial G_\nu}{\partial x_3}\right)dx_2dx_3$$
$$+ \sum_{\nu=1}^{n}\left(-a_{3\nu}\frac{\partial G_\nu}{\partial x_1} + a_{1\nu}\frac{\partial G_\nu}{\partial x_3}\right)dx_1dx_3$$
$$+ \sum_{\nu=1}^{n}\left(a_{2\nu}\frac{\partial G_\nu}{\partial x_1} - a_{1\nu}\frac{\partial G_\nu}{\partial x_2}\right)dx_1dx_2\Big\}.$$

Integrating by parts we obtain

$$I_2 = -\int\int_{\mathfrak{B}^2}\left(\nabla r^{-1}\times\sum_{\nu=1}^{n} G_\nu\mathfrak{a}_\nu\right)\cdot d\sigma$$

since the line integrals cancel each other. Suppose now that we have a harmonic vector $\mathfrak{H}(\mathfrak{X}) = \sum_{k=1}^{3} H^{(k)}(\mathfrak{X})\mathfrak{i}_k$, such that

$$\nabla H^{(k)} = \sum_{\nu=1}^{3}\nabla H^{(\nu)}\times\mathfrak{a}_\nu^{(k)}, \qquad k = 1, 2, 3. \tag{4.3}$$

(4.2) yields

$$H^{(k)}(\mathfrak{X}) = (1/4\pi)\int\int_{\mathfrak{B}^2}\left[H^{(k)}\nabla r^{-1} + \nabla r^{-1}\times\sum_{\nu=1}^{3} H^{(\nu)}\mathfrak{a}_\nu^{(k)}\right]\cdot d\sigma. \tag{4.4}$$

Thus:

$$\mathfrak{H}(\mathfrak{X}) = (1/4\pi)\int\int_{\mathfrak{B}^2}\Big[\mathfrak{H}(\nabla r^{-1}\cdot d\sigma)$$
$$+ \sum_{k=1}^{3}\left[\left(\nabla r^{-1}\times\sum_{\nu=1}^{3} H^{(\nu)}\mathfrak{a}_\nu^{(k)}\right)\cdot d\sigma\right]\mathfrak{i}_k\Big]$$
$$= (1/4\pi)\int\int_{\mathfrak{B}^2}\Big[\mathfrak{H}(\nabla r^{-1}\cdot d\sigma)$$
$$+ \sum_{\nu=1}^{3}\sum_{k=1}^{3}\left[H^{(\nu)}(d\sigma\times\nabla r^{-1})\cdot\mathfrak{a}_\nu^{(k)}\right]\mathfrak{i}_k\Big].$$

In particular if div $\mathfrak{H}=0$ and curl $\mathfrak{H}=0$ then the relation (4.3) is satisfied, and we can represent $\mathfrak{H}(\mathfrak{X})$ in the above form (4.4).[11]

REMARK. $d\sigma = \mathfrak{n}_1 do$, where $\mathfrak{n}_1$ is the unit vector in the direction of the interior normal, and do the area of the surface element. (We note that $dx_m dx_n$ means $+|dx_m dx_n|$ or $-|dx_m dx_n|$, the sign being chosen in such a manner that the above relation for $d\sigma$ is valid.)

(4.2) can also be obtained using the Stokes' formula. In fact

$$I_2 = \sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla G_\nu \times \mathfrak{a}_\nu) do = \sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla \times G_\nu \mathfrak{a}_\nu) do$$

since the $a_{k\nu}$ are constants. By Stokes' formula

$$\iint_{\mathfrak{B}^2} \mathfrak{n}_1(\nabla \times r^{-1} G_\nu \mathfrak{a}_\nu) do = \iint_{\mathfrak{B}^2} \mathfrak{n}_1 r^{-1}(\nabla \times G_\nu \mathfrak{a}_\nu) do + \iint_{\mathfrak{B}^2} \mathfrak{n}_1 G_\nu(\nabla \times r^{-1}\mathfrak{a}_\nu) do = 0,$$

since $\mathfrak{B}^2$ is a closed surface. Therefore

$$I_2 = -\sum_\nu \iint_{\mathfrak{B}^2} \mathfrak{n}_1 G_\nu(\nabla \times r^{-1}\mathfrak{a}_\nu) do.$$

We also note that formulas expressing the pressure on an immersed body (analogous to those of the two-dimensional theory) can be obtained using the fact that $\iint_{\mathfrak{B}^2}[\mathfrak{n}_1|\mathfrak{H}|^2 - 2\mathfrak{H}(\mathfrak{n}_1\mathfrak{H})]do$ does not change when $\mathfrak{B}^2$ is deformed, if curl $\mathfrak{H}=$ div $\mathfrak{H}=0$.

REMARK. The method presented here can obviously be applied if the components $H^{(n)}$ of a vector $\mathfrak{H}(\mathfrak{X})$ are connected by the relations

$$\sum_{n=1}^{3} \mathfrak{a}_{kn} \cdot \nabla H^{(n)} + \mathfrak{b}_k \cdot \mathfrak{H} + C_k = 0, \qquad k = 1, 2, 3,$$

where $\mathfrak{a}_{kn} = (a_{kn}^{(1)}\mathfrak{i}_1 + a_{kn}^{(2)}\mathfrak{i}_2 + a_{kn}^{(3)}\mathfrak{i}_3)$, $\mathfrak{b}_k = (b_k^{(1)}\mathfrak{i}_1 + b_k^{(2)}\mathfrak{i}_2 + b_k^{(3)}\mathfrak{i}_3)$, and the determinants $a_{kn}^{(q)} \neq 0$, for $n = 1, 2, 3$.

Analogous formulas are also valid if $H^{(k)}$ satisfy certain elliptic differential equations and if we introduce instead of r^{-1} a fundamental solution of this equation.

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[11] In the case where curl $\mathfrak{H}=$ div $\mathfrak{H}=0$ similar formulas were given by von Mises (an address presented at the New York meeting of the Society on April 4, 1942, by invitation of the Program Committee) and by Fulton and Rainich (Amer. J. Math. vol. 54 (1932) pp. 235–241).